Deriving the variance for the size of the nth generation of the branching process

Nguyen Duc Thanh

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1 Problem

Let μ and σ^2 be the mean and variance of the family-size distribution. Show that the variance of Z_n, the size of the nth generation of the branching process, is given by

$$var(Z_n) = \begin{cases} n\sigma^2 & \text{if } \mu = 1, \\ \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1. \end{cases}$$

2 Proof

Let C be the random variable corresponding to the number of children of an individual nomad, then

$$\mathbb{E}(C) = \mu \quad , \quad var(C) = \sigma^2.$$

Let G, G_n be the probability generating function of C and Z_n (for $n = 0, 1, ...$), respectively. In the lecture, we have proven that

$$\begin{aligned} &G_0(s) = s, \quad G_1(s) = G(s), \\ &G_n(s) = G_{n-1}(G(s)) = G(G(...G(s)...)), \\ &\mathbb{E}(Z_n) = G_n'(1) = \mu^n. \end{aligned}$$

For an integer-valued random variable X with probability generating function G_X, the kth derivative of $G_X(s)$ at $s = 1$ is given by

$$G_X^{(k)}(1) = \mathbb{E}(X[X-1]...[X-k+1]).$$

Then one can show that

$$\begin{aligned} \mathbb{E}(X^2) &= \mathbb{E}(X[X-1] + X) \\ &= \mathbb{E}(X[X-1]) + \mathbb{E}(X) \\ &= G_X''(1) + G_X'(1). \end{aligned}$$

Consider Z_n for some n

$$\mathbb{E}(Z_n^2) = G_n''(1) + G_n'(1) = G_n''(1) + \mu^n.$$

Since $G_n = G_{n-1} \circ G$, by the chain rule, one has

$$\begin{aligned} &G_n'(s) = G_{n-1}'(G(s))\, G'(s) \\ &G_n''(s) = G_{n-1}''(G(s))\, [G'(s)]^2 + G_{n-1}'(G(s))\, G''(s). \end{aligned}$$

By using $G(1) = 1$, $G'(1) = \mathbb{E}(C) = \mu$, and $G'_{n-1}(1) = \mu^{n-1}$,

$$\begin{aligned}
G''_n(1) &= \mu^2 G''_{n-1}(1) + \mu^{n-1} G''(s) \\
&= \mu^2[\mu^2 G''_{n-2}(1) + \mu^{n-2} G''(s)] + \mu^{n-1} G''(s) \\
&= \mu^4 G''_{n-2}(1) + \mu^{n-1}(1+\mu) G''(s) \\
&= \mu^4[\mu^2 G''_{n-3}(1) + \mu^{n-3} G''(s)] + \mu^{n-1}(1+\mu) G''(s) \\
&= \mu^6 G''_{n-3}(1) + \mu^{n-1}(1+\mu+\mu^2) G''(s) \\
&= ... \\
&= \mu^{2n} G''_0(1) + G''(s)\mu^{n-1} \sum_{i=0}^{n-1} \mu^i.
\end{aligned}$$

Since $G_0(s) = s$, its second derivative $G''_0(s) = 0$ for any s. Additionally, by evaluating the variance of C, one has

$$\begin{aligned}
var(C) = \sigma^2 &= \mathbb{E}(C^2) - \mathbb{E}(C)^2 \\
&= G''(1) + G'(1) - \mu^2 \\
&= G''(1) + \mu - \mu^2
\end{aligned}$$

$$\text{or} \quad G''(1) = \sigma^2 + \mu^2 - \mu.$$

Then

$$\begin{aligned}
\mathbb{E}(Z_n^2) &= (\sigma^2 + \mu^2 - \mu)\mu^{n-1} \sum_{i=0}^{n-1} \mu^i + \mu^n \\
&= \sigma^2 \mu^{n-1} \sum_{i=0}^{n-1} \mu^i + (\mu - 1)\mu^n \sum_{i=0}^{n-1} \mu^i + \mu^n.
\end{aligned}$$

Let $S = \sum_{i=0}^{n-1} \mu^i$, the variance of Z_n is given by

$$\begin{aligned}
var(Z_n) &= \mathbb{E}(Z_n^2) - \mathbb{E}(Z_n)^2 \\
&= \sigma^2 \mu^{n-1} S + (\mu - 1)\mu^n S + \mu^n - \mu^{2n} \\
&= \sigma^2 \mu^{n-1} S + (\mu - 1)\mu^n S + \mu^n(1 - \mu^n).
\end{aligned}$$

<u>If $\mu = 1$:</u>

One has

$$S = \sum_{i=0}^{n-1} 1^i = n.$$

Hence

$$\begin{aligned}
var(Z_n) &= n\sigma^2 + 0 + 1 - 1 \\
&= n\sigma^2.
\end{aligned}$$

<u>If $\mu \neq 1$:</u>

Evaluate $(\mu - 1)S$

$$\begin{aligned}
(\mu - 1)S &= \mu S - S \\
&= (\mu + \mu^2 + ... + \mu^n) - (1 + \mu + ... + \mu^{n-1}) \\
&= \mu^n - 1.
\end{aligned}$$

Then

$$S = \frac{\mu^n - 1}{\mu - 1} \qquad (\mu \neq 1).$$

Hence

$$\begin{aligned} var(Z_n) &= \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} + \mu^n(\mu^n - 1) + \mu^n(1 - \mu^n) \\ &= \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}. \end{aligned}$$

As a result, the variance of Z_n is given by

$$var(Z_n) = \begin{cases} n\sigma^2 & \text{if } \mu = 1, \\ \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1. \end{cases}$$

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