Bonferroni's inequality

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1 Problem

Prove that

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - n + \sum_{i=1}^{n} \mathbb{P}(A_i)$$

This is sometimes called Bonferroni's inequality.

2 Proof

The inequality can be proven by induction as follows

For $n = 1$:

$$\mathbb{P}(A_1) \geq 1 - 1 + \mathbb{P}(A_1) = \mathbb{P}(A_1)$$

which is always true.
Assume the inequality is true for some k, i.e.

$$\mathbb{P}\left(\bigcap_{i=1}^{k} A_i\right) \geq 1 - k + \sum_{i=1}^{k} \mathbb{P}(A_i) \tag{1}$$

Consider $\mathbb{P}\left(\bigcap_{i=1}^{k+1} A_i\right)$

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{k+1} A_i\right) &= \mathbb{P}\left[\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}\right] \\
&= \mathbb{P}\left(\bigcap_{i=1}^{k} A_i\right) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left[\left(\bigcap_{i=1}^{k} A_i\right) \cup A_{k+1}\right]
\end{aligned}$$

From equation (1), one has

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{k+1} A_i\right) &\geq 1 - k + \sum_{i=1}^{k} \mathbb{P}(A_i) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left[\left(\bigcap_{i=1}^{k} A_i\right) \cup A_{k+1}\right] \\
&\geq 1 - k + \sum_{i=1}^{k+1} \mathbb{P}(A_i) - \mathbb{P}\left[\left(\bigcap_{i=1}^{k} A_i\right) \cup A_{k+1}\right]
\end{aligned}$$

From the definition, the last term cannot exceed 1 so

$$\begin{aligned}\mathbb{P}\left(\bigcap_{i=1}^{k+1} A_i\right) &\geq 1 - k + \sum_{i=1}^{k+1} \mathbb{P}(A_i) - 1 \\ &\geq 1 - (k+1) + \sum_{i=1}^{k+1} \mathbb{P}(A_i)\end{aligned}$$

This means that if the statement is true for k, then it is also true for $k+1$.

By induction, we have proven the Bonferroni's inequality.

□