Solution to the Problem1, at P.19,[GW]

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1. A fair die is thrown n times. Show that the probability that there are an even number of sixes is $\frac{1}{2}[1+(\frac{2}{3})^n]$. For the purpose of this question, 0 is an even number.

We can use Mathmatical Induction to solve this. Here we define P(n) as the probability that there are an even number of sixes when a fair die is thrown n times.

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Obviously, $P(1) = \frac{5}{6}$, because the condition is satisfied when the result is $1, 2, 3, 4, or 5$. This is the case that $n = 1$.

I use here the M.I., Mathmatical Induction. If there are an even number of sixes after n times, at the next throw, it must be $1, 2, 3, 4, or 5$.

If there are an odd number of sixes after n times, then $P^c(n) = 1 - P(n) = 1 - \frac{1}{2}\left[1 + \left(\frac{2}{3}\right)^n\right]$, here $P^c(n)$ is defined as the probability that there are an odd number of sixes when a fair die is thrown n times. At the next throw, it must be 6 to satisfy the condition.

If $P(n) = \frac{1}{2}\left[1 + \left(\frac{2}{3}\right)^n\right]$ is true,

$$\begin{aligned} P(n+1) &= \left[1 - \frac{1}{2}\left[1 + \left(\frac{2}{3}\right)^n\right]\right] \times \frac{1}{6} + \frac{1}{2}\left[1 + \left(\frac{2}{3}\right)^n\right] \times \frac{5}{6} \\ &= \frac{1}{6} - \frac{1}{12}\left[1 + \left(\frac{2}{3}\right)^n\right] + \frac{5}{12}\left[1 + \left(\frac{2}{3}\right)^n\right] \\ &= \frac{6}{12} - \frac{1}{12}\left(\frac{2}{3}\right)^n + \frac{5}{12}\left(\frac{2}{3}\right)^n \\ &= \frac{1}{2} + \frac{1}{3}\left(\frac{2}{3}\right)^n \\ &= \frac{1}{2} + \frac{1}{2} \times \frac{2}{3}\left(\frac{2}{3}\right)^n \\ &= \frac{1}{2}\left[1 + \left(\frac{2}{3}\right)^{n+1}\right] \end{aligned}$$

By M.I., we reach the conclusion that $P(n) = \frac{1}{2}\left[1 + \left(\frac{2}{3}\right)^n\right]$.

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This formula, $P(n) = \frac{1}{2}\left[1 + \left(\frac{2}{3}\right)^n\right]$, is derived by solving $P(n+1) = \frac{5}{6}P(n) + \frac{1}{6}\left(1 - P(n)\right)$.