Exercise for Branching Process

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Question

If $\mu = \mathbb{E}(c)$ and $\sigma^2 = var(c)$,
need to show that

$$var(Z_n) = \begin{cases} \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} & \mu \neq 1 \\ n\sigma^2 & \mu = 1. \end{cases}$$

Proof

Let $\sigma_n^2 = var(Z_n)$, recall from Theorem 9.8 that $\mathbb{E}(Z_n) = \mu^n$.
Now,

$$\begin{aligned} G_n^{'}(s) =& G_{n-1}^{'}(G(s))G^{'}(s) \\ \implies G_n^{''}(s) =& [G_{n-1}^{'}(G(s))G^{'}(s)]^{'} \\ =& G_{n-1}^{''}(G(s))G^{'}(s)^2 + G_{n-1}^{'}(G(s))G^{''}(s). \end{aligned}$$

Then, consider the value of the probability generating function $G_n(s)$:
Knowing that $G_n(s) := \sum_{k=0}^{n} s^k p_k$, where $p_k = \mathbb{P}(Z_n = k)$.

Then,

$$G_n^{'}(s) = \sum_{k=0}^{n} k s^k p_k,$$

$$G_n^{''}(s) = \sum_{k=0}^{n} k(k-1) s^k p_k.$$

Let s=1 and n=1, we get

$$\begin{aligned} G(1) =& \sum_{k=0}^{n} p_k = 1 \\ G^{'}(1) =& \sum_{k=0}^{n} k p_k = \mu \\ G^{''}(1) =& \sum_{k=0}^{n} k(k-1) p_k \\ =& \sum_{k=0}^{n} (k^2 p_k - k p_k) \\ =& \sum_{k=0}^{n} k^2 p_k - (\sum_{k=0}^{n} k p_k)^2 - \sum_{k=0}^{n} k p_k + (\sum_{k=0}^{n} k p_k)^2 \\ =& \sigma^2 - \mu + \mu^2 \\ \implies \sigma^2 =& G^{''}(1) + \mu - \mu^2. \end{aligned}$$

Also, apply Theorem 9.8, and the equations, we get

$$\sigma_n^2 = G_n^{''}(1) + \mu^n - \mu^{2n}.$$

Then we have

$$G_n^{'}(1) = \mu^n \quad (1)$$
$$G_n^{''}(1) = \sigma_n^2 - \mu^n + \mu^{2n}. \quad (2)$$

Then

$$G_{n-1}^{'}(1) = \mu^{n-1} \quad (3)$$
$$G_{n-1}^{''}(1) = \sigma_{n-1}^2 - \mu^{n-1} + \mu^{2(n-1)}. \quad (4)$$

We derived from above that

$$G_n^{''}(s) = G_{n-1}^{''}(G(s))G^{'}(s)^2 + G_{n-1}^{'}(G(s))G^{''}(s).$$

Substitute (1)(2)(3)(4) into the equation when s=1,

$$\begin{aligned} G_n^{''}(1) &= G_{n-1}^{''}(G(1))G^{'}(1)^2 + G_{n-1}^{'}(G(1))G^{''}(1) \\ \sigma_n^2 - \mu^n + \mu^{2n} &= G_{n-1}^{''}(1)\mu^2 + G_{n-1}^{'}(1)(\sigma^2 - \mu + \mu^2) \\ &= (\sigma_{n-1}^2 - \mu^{n-1} + \mu^{2(n-1)})\mu^2 + \mu^{n-1}(\sigma^2 - \mu + \mu^2) \\ &= \mu^2\sigma_{n-1}^2 - \mu^{n+1} + \mu^{2n} + \sigma^2\mu^{n-1} - \mu^n + \mu^{n+1}, \end{aligned}$$

We derive

$$\sigma_n^2 = \mu^2\sigma_{n-1}^2 + \sigma^2\mu^{n-1}.$$

Since we have

$$\begin{aligned} \sigma_1^2 &= \sigma^2, \\ \sigma_2^2 &= \mu^2\sigma_1^2 + \sigma^2\mu = \mu\sigma^2(\mu+1), \\ \sigma_3^2 &= \mu^2\sigma_2^2 + \sigma^2\mu^2 \\ &= \mu^2\mu\sigma^2(\mu+1) + \sigma^2\mu^2 \\ &= \mu^2\sigma^2(\mu^2+\mu+1), \end{aligned}$$

and so on...
Therefore, we find that

$$\begin{aligned} varZ_n = \sigma_n^2 &= \mu^{n-1}\sigma^2 \sum_{i=0}^{n-1} \mu^i \\ &= \begin{cases} \sigma^2\mu^{n-1}\frac{\mu^n-1}{\mu-1} & \mu \neq 1 \\ n\sigma^2 & \mu = 1. \quad \square \end{cases} \end{aligned}$$