SML: Probabilities Exercise problem for 1st lecture PENG Qi 061901856

○ Proof of Bool's ~~equation~~ inequality: $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \leqslant \sum_{i=1}^{n} \mathbb{P}(A_i)$

⇒ Prove by induction: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be probability space.

① For $n=1$: $A_1 \in \mathcal{F}$, $\mathbb{P}(A_1) \leqslant \mathbb{P}(A_1)$ satisfied.

② Assume that for n events: $A_i \in \mathcal{F}$ for any i $(1 \leqslant i \leqslant n)$, $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \leqslant \sum_{i=1}^{n} \mathbb{P}(A_i)$

Set $A := \bigcup_{i=1}^{n} A_i \Leftrightarrow \mathbb{P}(\bigcup_{i=1}^{n} A_i) = \mathbb{P}(A)$.

We need to show that $\mathbb{P}(A \cup A_{n+1}) \leqslant \sum_{i=1}^{n+1} \mathbb{P}(A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i) + \mathbb{P}(A_{n+1})$

○ Then introduce the property in Page 8. GW:

$\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B)$, for $A, B \in \mathcal{F}$

$\Rightarrow \mathbb{P}(A \cup A_{n+1}) = \mathbb{P}(A) + \mathbb{P}(A_{n+1}) - \mathbb{P}(A \cap A_{n+1})$.

○ Then by induction, $\mathbb{P}(A) \leqslant \sum_{i=1}^{n} \mathbb{P}(A_i) \Rightarrow \mathbb{P}(A \cup A_{n+1}) \leqslant \sum_{i=1}^{n} \mathbb{P}(A_i) + \mathbb{P}(A_{n+1}) - \mathbb{P}(A \cap A_{n+1})$

$\Leftrightarrow \mathbb{P}(\bigcup_{i=1}^{n+1} A_i) \leqslant \sum_{i=1}^{n} \mathbb{P}(A_i) + \mathbb{P}(A_{n+1}) - \mathbb{P}(A \cap A_{n+1})$

○ Since we know that $\mathbb{P}(A \cap A_{n+1}) \geqslant 0$

$\Rightarrow \sum_{i=1}^{n} \mathbb{P}(A_i) + \mathbb{P}(A_{n+1}) - \mathbb{P}(A \cap A_{n+1}) \leqslant \sum_{i=1}^{n} \mathbb{P}(A_i) + \mathbb{P}(A_{n+1})$

$\Rightarrow \mathbb{P}(\bigcup_{i=1}^{n+1} A_i) \leqslant \sum_{i=1}^{n} \mathbb{P}(A_i) + \mathbb{P}(A_{n+1})$. □

Therefore, we proved the Bool's inequality by induction.