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On the gamma distribution

[Related to exercise 7.59a and problem 7.5]

The gamma distribution with parameter $w > 0$ and $\lambda > 0$ has the density function:

$$f(x) = \begin{cases} \frac{1}{\Gamma(w)} \lambda^{w} x^{w-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where $\Gamma(w)$ is the gamma function, defined by:

$$\Gamma(w) = \int_0^\infty x^{w-1} e^{-x} dx$$

Let X be a random variable having the gamma distribution with parameters w and λ. The moment generating function of X is given by:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \qquad \text{for } |t| < \delta \text{ for some } \delta > 0$$

$$= \frac{\lambda^w}{\Gamma(w)} \int_0^\infty e^{tx} x^{w-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^w}{\Gamma(w)} \int_0^\infty e^{-(\lambda - t)x} x^{w-1} dx$$

Set $y = (\lambda - t)x$ and we consider $|t| < \lambda$. Hence:

$$M_X(t) = \frac{\lambda^w}{\Gamma(w)} \int_0^\infty e^{-y} \left(\frac{y}{\lambda - t}\right)^{w-1} d\left(\frac{y}{\lambda - t}\right) \qquad \text{for } |t| < \lambda$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^w \frac{1}{\Gamma(w)} \underbrace{\int_0^\infty e^{-y} y^{w-1} dy}_{= \Gamma(w)}$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^w$$

The term on the right-hand side is finite for $|t| < \lambda$, so we have the moment generating function of X having the gamma distribution with parameters w and λ is:

$$\boxed{M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^w \qquad \text{for } |t| < \lambda}$$

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Let X and Y be independent random variables, X having the gamma distribution with parameters s and λ, and Y having the gamma distribution with parameters t and λ. The moment generating functions of X and Y are:

$$M_X(r) = \left(\frac{\lambda}{\lambda - r}\right)^s \qquad \text{for } |r| < \lambda$$

$$M_Y(r) = \left(\frac{\lambda}{\lambda - r}\right)^t \qquad \text{for } |r| < \lambda$$

X, Y are independent so the moment generating function of $X+Y$ is given by:

$$\begin{aligned} M_{X+Y}(r) &= M_X(r)\, M_Y(r) \qquad \text{for } |r| < \lambda \\ &= \left(\frac{\lambda}{\lambda - r}\right)^s \left(\frac{\lambda}{\lambda - r}\right)^t \\ &= \left(\frac{\lambda}{\lambda - r}\right)^{s+t} \end{aligned}$$

This is the same as the moment generating function of a random variable having gamma distribution with parameters $s+t$ and λ. By the uniqueness theorem (for moment generating functions), $X+Y$ has the gamma distribution with parameters $s+t$ and λ.