Introduction to Probability

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Report

Problem 1 (GW- Page 58) Let X have probability function $G_X(s)$ and let $u_n = \mathbb{P}(X > n)$. Show that the generating function $U(s)$ of the sequence $u_0, u_1, \dots$ satisfies $(1-s)U(s) = 1 - G_X(s)$

Set $P_n := \mathbb{P}(X = n)$, then the probability generating function of X:

$$G_X(s) = \sum_{n=0}^{\infty} P_n s^n = \sum_{n=0}^{\infty} \mathbb{P}(X = n) s^n \qquad (n \in \mathbb{N})$$

One has $u_n = \mathbb{P}(X > n)$ and $u_{n-1} = \mathbb{P}(X > n-1)$, then:

$$P_n = \mathbb{P}(X = n) = \mathbb{P}(n-1 < X \leq n) = \mathbb{P}(X > n-1) - \mathbb{P}(X > n)$$

$$\Rightarrow P_n = u_{n-1} - u_n \qquad (\text{for all } n \geq 1)$$

And for $n = 0$, $P_0 = \mathbb{P}(X = 0) = 1 - \mathbb{P}(X > 0) = 1 - u_0$

$U(s)$ is a generating function of the sequence $u_0, u_1, \dots$ then:

$$U(s) = \sum_{n=0}^{\infty} u_n s^n$$

One has

$$\begin{aligned}
1 - G_X(s) &= 1 - P_0 - \sum_{n=1}^{\infty} (u_{n-1} - u_n) s^n \\
&= 1 - (1 - u_0) - \sum_{n=1}^{\infty} u_{n-1} s^n + \sum_{n=1}^{\infty} u_n s^n \\
&= -s \sum_{n=0}^{\infty} u_n s^n + \left(u_0 s^0 + \sum_{n=1}^{\infty} u_n s^n\right) \\
&= -s\,U(s) + U(s) \\
&= U(s)(1 - s)
\end{aligned}$$

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