Alternative form of Chebyshev's inequality and rolling of a fair dice

LIN Guozhang (061801907)

This report uses the alternative form of Chebyshev's inequality to solve Exercise 8.21 of the textbook, *Probability, an introduction.*

According to page 71 of the lecture notes, if Y is a random variable with $\mathbb{E}(Y^2) < \infty$, then

$$\mathbb{P}(|Y| \geq a) \leq \frac{1}{a^2}\mathbb{E}(Y^2), \text{ where } a > 0.$$

Let $Y = X - \mathbb{E}(X)$. Then

$$\begin{aligned}\mathbb{E}(Y^2) &= \mathbb{E}\{[X - \mathbb{E}(X)]^2\} \\ &= \text{var}(X) < \infty\end{aligned}$$

Thus, if X is a random variable with finite variance,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a) \leq \frac{1}{a^2}\text{var}(X), \text{ where } a > 0. \tag{1}$$

(1) is the alternative form of Chebyshev's inequality.

Exercise 8.21 Use Chebyshev's inequality to show that the probability that in n throws of a fair dice the number of sixes lie between $\frac{1}{6}n - \sqrt{n}$ and $\frac{1}{6}n + \sqrt{n}$ is at least $\frac{31}{36}$.
Solution
For $i = 1, ..., n$, define Z_i as follows.

$$Z_i = \begin{cases} 1, & \text{if the dice gives 6 for the i-th roll.} \\ 0, & \text{otherwise.} \end{cases}$$

For a fair dice, $\mathbb{P}(Z_i = 1) = \frac{1}{6}$ for $i = 1, ..., n$, and $Z_1, ..., Z_n$ are independent.
Thus, $Z_i \sim B(\frac{1}{6})$ for $i = 1, ..., n$, and

$$X := \sum_{i=1}^{n} Z_i \sim B(n, \frac{1}{6})$$

For a binomial random variable, its mean is np, and its variance is $np(1-p)$.
Thus $\mathbb{E}(X) = \frac{n}{6}$, $\text{var}(X) = n\frac{1}{6}(1 - \frac{1}{6}) = \frac{5n}{36}$.
Then by (1),

$$\mathbb{P}(|X - \frac{n}{6}| \geq a) \leq \frac{1}{a^2}\frac{5n}{36}, \text{ where } a > 0.$$

Let $a = \sqrt{n}$. Then,

$$\mathbb{P}(|X - \frac{n}{6}| \geq \sqrt{n}) \leq \frac{1}{n}\frac{5n}{36}$$

Thus,

$$\mathbb{P}(X \geq \frac{n}{6} + \sqrt{n} \text{ or } X \leq \frac{n}{6} - \sqrt{n}) \leq \frac{5}{36}$$

$$\begin{aligned}\mathbb{P}(\frac{n}{6} - \sqrt{n} < X < \frac{n}{6} - \sqrt{n}) &= 1 - \mathbb{P}(X \geq \frac{n}{6} + \sqrt{n} \text{ or } X \leq \frac{n}{6} - \sqrt{n}) \\ &\geq 1 - \frac{5}{36} \\ &= \frac{31}{36}\end{aligned}$$

Therefore, we have shown that the probability that in n throws of a fair dice the number of sixes lie between $\frac{1}{6}n - \sqrt{n}$ and $\frac{1}{6}n + \sqrt{n}$ is at least $\frac{31}{36}$.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.