Properties Related to MGF

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This report is to prove the three properties related to moment generating functions listed on page 64 of the lecture notes, which are

1. $M_{aX+b}(t) = \exp(tb)M_X(at)$.
2. If X and Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$, which is true on the common interval where both mgf's exist.
3. If $X \sim N(\mu, \sigma^2)$, then $M_X(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$.

Proof of property 1
Let $Y := aX + b$.

$$\begin{aligned} M_Y(t) &= \mathbb{E}[\exp(tY)] \\ &= \mathbb{E}[\exp(atX + tb)] \\ &= \mathbb{E}[\exp(atX)\exp(tb)] \end{aligned}$$

Since there is no random variable in $\exp(tb)$, it can be taken out of the expectation.

$$\begin{aligned} M_Y(t) &= \exp(tb)\mathbb{E}[\exp(atX)] \\ &= \exp(tb)M_X(at) \end{aligned}$$

Q.E.D

Proof of property 2

$$\begin{aligned} M_{X+Y}(t) &= \mathbb{E}\{\exp[t(X+Y)]\} \\ &= \mathbb{E}[\exp(tX + tY)] \\ &= \mathbb{E}[\exp(tX)\exp(tY)] \end{aligned}$$

According to equation (6.64) in the textbook, for independent X and Y,

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

Thus,

$$\begin{aligned} M_{X+Y}(t) &= \mathbb{E}[\exp(tX)]\mathbb{E}[\exp(tY)] \\ &= M_X(t)M_Y(t) \end{aligned}$$

Q.E.D.

Proof of property 3
If $X \sim N(\mu, \sigma^2)$, then the pdf of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2]$$

The moment generating function of X is

$$\begin{aligned}
M_X(t) &= \mathbb{E}[\exp(tX)] \\
&= \int_{-\infty}^{\infty} f_X(x)\exp(tx)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2]\exp(tx)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2 + tx]dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2 - 2\sigma^2 tx)]dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}[x^2 - 2x(\mu+\sigma^2 t)] - \frac{\mu^2}{2\sigma^2}\}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}[x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2 - (\mu+\sigma^2 t)^2] - \frac{\mu^2}{2\sigma^2}\}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}[x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2] + \frac{(\mu+\sigma^2 t)^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}\}dx \\
&= \exp[\frac{(\mu+\sigma^2 t)^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}[x - (\mu+\sigma^2 t)]^2\}dx \\
&= \exp(\frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}y^2)dy, \text{ where } y := x - (\mu+\sigma^2 t). \\
&= \exp(\mu t + \frac{\sigma^2 t^2}{2}) \int_{-\infty}^{\infty} f_Y(y)dy, \text{ if } Y \sim N(0,\sigma^2). \\
&= \exp(\mu t + \frac{\sigma^2 t^2}{2}) \cdot 1, \text{ by the property of pdf.} \\
&= \exp(\mu t + \frac{\sigma^2 t^2}{2})
\end{aligned}$$

Q.E.D.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.