Chi-Squared Distribution

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This report is to explore the chi-squared distribution. It is partly related to Exercise 5.55 of the textbook, *Probability: an introduction.*

According to page 69 of the textbook, the chi-squared distribution with n degrees of freedom (sometimes written as χ_n^2) has density function

$$f(y) = \begin{cases} \frac{1}{2\Gamma(n/2)}(y/2)^{n/2-1}e^{-y/2} & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{1}$$

A comparison of (1) in this report with (1.1) in Lin Guozhang's report, *Gamma Function and Gamma Distribution* shows that χ_n^2 is the same as $\Gamma(n/2, 1/2)$, but the chi-squared distribution is important due to its common occurrence in statistics.

Properties of the chi-squared distribution

1. If $X \sim N(0,1)$, then $X^2 \sim \chi_1^2$.
2. If X_1, ..., X_k are independent, and $X_i \sim \chi_{n_i}^2$, where $i = 1$, 2, ... k, then

$$\Sigma_{i=1}^k X_i \sim \chi^2_{\sum_{i=1}^k n_i}$$

3. If $X_1, ..., X_n$ are independent, and $X_i \sim N(0,1)$, where $i = 1$, 2, ... n, then

$$\Sigma_{i=1}^n X_i^2 \sim \chi_n^2$$

Proof of property 1

To prove property 1, the result on page 45 of the lecture notes for SML is used, which is as follows.
If X is a continuous random variable with probability density function f_X and $Y := X^2$, then the probability density function of Y is

$$f_Y(y) = \begin{cases} \frac{1}{2}\frac{1}{\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})) & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{2}$$

Now, according to (5.38) of the textbook, if $X \sim N(0,1)$, then for $x \in \mathbb{R}$,

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2) \tag{3}$$

By (2) and (3),

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}}e^{-y/2} & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{4}$$

When n=1, (1) becomes (4). (For the calculation of $\Gamma(1/2)$, please refer to Lin Guozhang's report, *Gamma Function and Gamma Distribution.*) With the same pdf, $Y = X^2 \sim \chi_1^2$.

Q.E.D.

Proof of property 2

To prove property 2, one result in Lin Guozhang's report *Gamma Function and Gamma Distribution* is used, which is as follows.
If $X_1, ..., X_k$ are independent, and $X_i \sim \Gamma(w_i, \lambda)$, where $i = 1$, 2, ... k, then

$$\Sigma_{i=1}^k X_i \sim \Gamma(\Sigma_{i=1}^k w_i, \lambda)$$

Let $w_i = n_i/2$ and $\lambda = 1/2$. Then it becomes as follows.
If $X_1, ..., X_k$ are independent, and $X_i \sim \Gamma(n_i/2, 1/2)$, where $i = 1$, 2, ... k, then

$$\Sigma_{i=1}^k X_i \sim \Gamma(\Sigma_{i=1}^k n_i/2, 1/2)$$

Since $\chi_{n_i}^2$ is the same as $\Gamma(n_i/2, 1/2)$, it can be rewritten as follows.
If $X_1, ..., X_k$ are independent, and $X_i \sim \chi_{n_i}^2$, where $i = 1$, 2, ... k, then

$$\Sigma_{i=1}^k X_i \sim \chi_{\Sigma_{i=1}^k n_i}^2$$

Q.E.D.

Proof of property 3

Let $Y_i := X_i^2$. Then by property 1, $Y_i \sim \chi_1^2$.
By property 2,

$$\Sigma_{i=1}^n Y_i \sim \chi_{\sum_{i=1}^n 1}^2$$

That is,

$$\Sigma_{i=1}^n X_i^2 \sim \chi_n^2$$

Q.E.D.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.
- *Gamma Function and Gamma Distribution* by Lin Guozhang.