Lack-of-memory property of geometric variables

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This report solves Exercise 5 in Chapter 2 of *Probability: an introduction*.

Claim 1
If X has the geometric distribution with parameter p, then it has the property of lack-of-memory, that is, for $m, n \in \mathbb{N}$,

$$\mathbb{P}(X > m + n | X > m) = \mathbb{P}(X > n).$$

Proof of claim 1
If X has the geometric distribution with parameter p, then its probability mass function is

$$\pi_k = p(1-p)^{k-1} \text{ , } k \in \mathbb{N}_+.$$

$$\begin{aligned}
\mathbb{P}(X > m + n | X > m) &= \frac{\mathbb{P}(\{X > m + n\} \cap \{X > m\})}{\mathbb{P}(X > m)} \\
&= \frac{\mathbb{P}(X > m + n)}{\mathbb{P}(X > m)}, \text{ since } \{X > m + n\} \subseteq \{X > m\}. \\
&= \frac{\sum_{i=m+n+1}^{\infty} \pi_i}{\sum_{i=m+1}^{\infty} \pi_i}, \text{ by the definition of the probability mass function.} \\
&= \frac{\sum_{i=m+n+1}^{\infty} p(1-p)^{i-1}}{\sum_{i=m+1}^{\infty} p(1-p)^{i-1}} \\
&= \frac{(1-p)^{m+n}}{(1-p)^m}, \text{ by the formula for geometric series.} \\
&= (1-p)^n \\
&= \sum_{i=n}^{\infty} p(1-p)^i, \text{ again, by the formula for geometric series.} \\
&= \sum_{i=n}^{\infty} \pi_{i+1} \\
&= \sum_{i=n+1}^{\infty} \pi_i \\
&= \mathbb{P}(X > n)
\end{aligned}$$

Q.E.D.

Claim 2
If a random variable X satisfies $\mathbb{P}(X > m + n | X > m) = \mathbb{P}(X > n)$, and $X \in \mathbb{N}_+$, m, $n \in \mathbb{N}$, then X is a geometric random variable.

Proof of claim 2

$$\begin{aligned}\mathbb{P}(X > m+n|X > m) &= \mathbb{P}(X > n)\\ \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > m)} &= \mathbb{P}(X > n)\\ \mathbb{P}(X > m+n) &= \mathbb{P}(X > m)\mathbb{P}(X > n)\end{aligned} \tag{1}$$

Define $\alpha_k := \mathbb{P}(X > k)$, $k \in \mathbb{N}$. Then (1) becomes

$$\alpha_{m+n} = \alpha_m \alpha_n \tag{2}$$

From (2), for $j \in \mathbb{N}_+$,

$$\alpha_j = \alpha_1^j \tag{3}$$

Since by mathematical induction,

$$\alpha_1 = \alpha_1^1 \tag{4}$$

Assume $\alpha_k = \alpha_1^k$, $k \in \mathbb{N}_+$. Then from (2),

$$\begin{aligned}\alpha_{k+1} &= \alpha_k \alpha_1\\ &= \alpha_1^k \alpha_1\\ &= \alpha_1^{k+1}\end{aligned} \tag{5}$$

By (4) and (5), (3) is verified.

The probability mass function of X is

$$\begin{aligned}\pi_i &= \mathbb{P}(X = i), \text{ where } i \in \mathbb{N}_+\\ &= \mathbb{P}(X > i-1) - \mathbb{P}(X > i)\\ &= \alpha_{i-1} - \alpha_i\\ &= \alpha_1^{i-1} - \alpha_1^i, \text{ according to (3).}\\ &= \alpha_1^{i-1}(1-\alpha_1)\end{aligned} \tag{6}$$

Define $p := \mathbb{P}(X = 1)$, then

$$\begin{aligned}p &= \mathbb{P}(X > 0) - \mathbb{P}(X > 1)\\ p &= 1 - \mathbb{P}(X > 1), \text{ since } X \in \mathbb{N}_+\\ p &= 1 - \alpha_1\\ \alpha_1 &= 1 - p\end{aligned} \tag{7}$$

By (6) and (7),

$$\pi_i = p(1-p)^{i-1},\ i \in \mathbb{N}_+$$

Thus, X obeys the geometric distribution of parameter p.

Q.E.D.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.