Moment generating function of Poisson distribution

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This report proves that the mgf of the Poisson distribution is

$$M(t) = \exp[\lambda(e^t - 1)].$$

One definition of the exponential function will be used in this report, which is the following.

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{1}$$

The radius of convergence of this power series is infinite.

Assume a random variable $X \sim P(\lambda)$, then, according to page 17 of the lecture notes, the pdf of X is

$$p_k = \frac{1}{k!}\lambda^k \exp(-\lambda), \text{ for } k\text{=0, 1, ...}$$

Then, the mgf of X is

$$\begin{aligned}
M(t) &= \mathbb{E}[\exp(tX)] \\
&= \sum_{k=0}^{\infty} \exp(tk) p_k \\
&= \sum_{k=0}^{\infty} \exp(tk) \frac{1}{k!} \lambda^k \exp(-\lambda) \\
&= \exp(-\lambda) \sum_{k=0}^{\infty} \frac{(e^t\lambda)^k}{k!} \\
&= \exp(-\lambda)\exp(e^t\lambda), \text{ according to (1);} \\
&= \exp[\lambda(e^t - 1)].
\end{aligned}$$

Q.E.D.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.
- `https://en.wikipedia.org/wiki/Exponential_function`