A transformation of gamma distribution tends to normal

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This report solves Exercise 8.33 of the textbook, *Probability, an introduction.*

According to page 151 of the textbook, the continuity theorem states that if Z_1, Z_2, ... are a sequence of random variables with mgf M_1, M_2, ... and suppose that, as $n \to \infty$,

$$M_n(t) \to \exp(\frac{1}{2}t^2).$$

Then

$$\mathbb{P}(Z_n \leq x) \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}u^2)du, \text{ for } x \in \mathbb{R}.$$

That theorem will be used to solve the exercise.

Exercise 8.33 For $n = 1, 2, ...,$ let $X_n \sim \Gamma(n, 1)$.
Claim 1: Mgf of $Z_n = \frac{X_n - n}{\sqrt{n}}$ is

$$M_n(t) = \exp(-t\sqrt{n})(1 - \frac{t}{\sqrt{n}})^{-n}.$$

Claim 2: As $n \to \infty$,

$$\mathbb{P}(Z_n \leq x) \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}u^2)du, \text{ for } x \in \mathbb{R}.$$

Proof of claim 1
As proved in *On the Gamma Distribution* by by Matsumoto Kosuke and Nguyen Duc Thanh, the mgf of a random variable $Y \sim \Gamma(w, \lambda)$ is

$$M_Y(t) = (\frac{\lambda}{\lambda - t})^w, \text{ for } t < \lambda.$$

Thus,

$$M_{X_n}(t) = (\frac{1}{1-t})^n, \text{ for } t < 1.$$

As proved in *Properties of the moment generating function* by Lin Guozhang,

$$M_{aX+b}(t) = \exp(tb)M_X(at).$$

Then,

$$\begin{aligned} M_n(t) &= M_{Z_n}(t) \\ &= M_{\frac{1}{\sqrt{n}}X_n - \sqrt{n}}(t) \\ &= \exp(-\sqrt{n}t)M_{X_n}(\frac{1}{\sqrt{n}}t) \\ &= \exp(-\sqrt{n}t)(\frac{1}{1 - \frac{1}{\sqrt{n}}t})^n \\ &= \exp(-\sqrt{n}t)(1 - \frac{1}{\sqrt{n}}t)^{-n}. \end{aligned}$$

Q.E.D.

Proof of claim 2
By the continuity theorem, claim 2 holds if

$$\lim_{n\to\infty} M_n(t) = \exp(\frac{1}{2}t^2). \tag{1}$$

By claim 1,

$$M_n(t) = \exp(-t\sqrt{n})(1-\frac{t}{\sqrt{n}})^{-n}.$$

Take logarithm of both sides and rearrange,

$$\ln M_n(t) = -t\sqrt{n} - n\ln(1-\frac{t}{\sqrt{n}}).$$

Let $x := \frac{1}{\sqrt{n}}$. Then

$$\ln M_n(t) = -\frac{t}{x} - \frac{1}{x^2}\ln(1-tx). \tag{2}$$

The Taylor series of $\ln(1-s)$ is as follows.

$$\ln(1-s) = -s - \frac{s^2}{2} - \frac{s^3}{3} - ..., \text{ for } s \in [-1,1).$$

Thus,

$$\ln(1-tx) = -tx - \frac{(tx)^2}{2} - \frac{(tx)^3}{3} - ..., \text{ for } tx \in [-1,1).$$

Plug it into (2). Then, for $tx \in [-1,1)$,

$$\begin{aligned}
\ln M_n(t) &= -\frac{t}{x} - \frac{1}{x^2}(-tx - \frac{(tx)^2}{2} - \frac{(tx)^3}{3} - ...) \\
&= -\frac{t}{x} + \frac{t}{x} + \frac{t^2}{2} + \frac{t^3x}{3} + ... \\
&= \frac{t^2}{2} + \frac{t^3x}{3} + ...
\end{aligned}$$

By the definition of x, as $n\to\infty$, $x\to 0$.
Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\ln M_n(t) &= \lim_{x\to 0}\ln M_n(t) \\
&= \lim_{x\to 0}\frac{t^2}{2} + \frac{t^3x}{3} + ... \\
&= \frac{t^2}{2} + 0 + ... \\
&= \frac{t^2}{2}
\end{aligned}$$

Since $\exp(\cdot)$ is a continuous function,

$$\begin{aligned}
\lim_{n\to\infty}\exp(\ln M_n(t)) &= \exp(\lim_{n\to\infty}\ln M_n(t)) \\
&= \exp(\frac{t^2}{2}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} M_n(t) &= \lim_{n\to\infty}\exp(\ln M_n(t)) \\
&= \exp(\frac{t^2}{2}).
\end{aligned}$$

Thus, (1) holds, and claim 2 is proved.

Q.E.D.

References

- *Probability, an introduction* from Grimmett and Welsh
- Lecture notes for SML: Probability by Richard, S.
- *On the Gamma Distribution* by Matsumoto Kosuke and Nguyen Duc Thanh
- `https://en.wikipedia.org/wiki/Taylor_series#Natural_logarithm`