Sum of Poisson-distributed random variables

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This report is related to Exercise 3.29 of *Probability: an introduction*, the textbook for the course of SML: probability.

Claim
If X_1, ..., X_n are independent random variables, ($n \in \mathbb{N}_+$), and X_i obeys Poisson distribution with parameter λ_i, $i \in [1, n]$, then $Y := \sum_{i=1}^{n} X_i$ obeys Poisson distribution with parameter $\sum_{i=1}^{n} \lambda_i$.

Proof
The following proves the claim by mathematical induction.
Base step:
By convolution formula,

$$p_{X_1+X_2}(y) = \sum_x p_{X_1}(x) p_{X_2}(y-x) \tag{1}$$

The probability mass function of a Poisson distributed random variable X with parameter λ is

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!},\ k \in \mathbb{N}$$

Thus, in (1), $x \in \mathbb{N}$ and $y - x \in \mathbb{N}$, that is, $0 \leq x \leq y$.

(1) becomes

$$\begin{aligned} p_{X_1+X_2}(y) &= \sum_{x=0}^{y} \frac{\lambda_1^x e^{-\lambda_1}}{x!} \frac{\lambda_2^{y-x} e^{-\lambda_2}}{(y-x)!} \\ &= \sum_{x=0}^{y} \frac{\lambda_1^x e^{-\lambda_1}}{x!} \frac{\lambda_2^{y-x} e^{-\lambda_2}}{(y-x)!} \frac{y!}{y!} \\ &= \frac{e^{-\lambda_1-\lambda_2}}{y!} \sum_{x=0}^{y} \frac{y!}{(y-x)!x!} \lambda_1^x \lambda_2^{y-x} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{y!} (\lambda_1+\lambda_2)^y, \text{ according to the binomial formula.} \end{aligned} \tag{2}$$

(2) is the probability mass function of a Poisson distributed random variable with parameter $\lambda_1 + \lambda_2$.
Thus, $X_1 + X_2$ obeys Poisson distribution with parameter $\lambda_1 + \lambda_2$.

Induction step:
Now, assume that if X_1, ..., X_{j-1} are independent random variables (j =3, 4, ...) and X_i obeys Poisson distribution with parameter λ_i, i =1, 2, ..., $j-1$ then $X' := \sum_{i=1}^{j-1} X_i$ obeys Poisson distribution with parameter $\sum_{i=1}^{j-1} \lambda_i$.

If X_j is a random variable independent of X' with parameter λ_j, then by (2),

$$p_{X'+X_j}(y) = \frac{e^{-(\sum_{i=1}^{j-1} \lambda_i + \lambda_j)}}{y!} (\sum_{i=1}^{j-1} \lambda_i + \lambda_j)^y$$

$$p_{\sum_{i=1}^{j} X_i}(y) = \frac{e^{-(\sum_{i=1}^{j} \lambda_i)}}{y!} (\sum_{i=1}^{j} \lambda_i)^y \tag{3}$$

(3) is the probability mass function of a Poisson distributed random variable with parameter $\sum_{i=1}^{j} \lambda_i$.
Thus, $Y := \sum_{i=1}^{j} X_i$ obeys Poisson distribution with parameter $\sum_{i=1}^{j} \lambda_i$.

By 3.1 and 3.2, the claim is proved.

Q.E.D.

Remark

The claim would be false without the condition of independence. For example, if X_1 and X_2 are Poisson distributed random variables, and $X_1 = X_2$, which implies that their values are dependent on each other, then $Y := X_1 + X_2 = 2X_1$ is not Poisson distributed because $\mathbb{P}$(Y is an odd number)$= 0$, violating the probability mass function of a Poisson distributed random variable.

References

- Lecture notes for SML: Probability by Richard, S.
- *Probability, an introduction* from Grimmett and Welsh