SMC: Introduction to Probability

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Exercises 10.4 from Grimmett and Welsh's Text.

In the following, $S_0, S_1, \ldots$ is a random walk on the integers with $p\ (=1-q)$ is probability that any given step is to the right.

Find the mean and variance of S_n when $S_0 = 0$

$$S_n = S_0 + \sum_{j=1}^{n} X_j$$

Mean : $E(S_n) = E(S_0 + \sum_{j=1}^{n} X_j)$ linearity

$$= E(S_0) + n\,E(X_1)$$

$$= 0 + n[(1)(p) + (-1)(1-p)]$$

$$\therefore E(S_n) = n(2p-1)$$

Variance : $E(S_n^2) = E((S_0 + \sum_{j=1}^{n} X_j)^2)$

$$= E\left(\left(\sum_{j=1}^{n} X_j\right)^2\right)$$

$$= E\left(\sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} X_j X_i + \sum_{j}^{n} X_j^2\right)$$

$$= E\left(\sum_{j=1}^{n}\sum_{\substack{i=1 \\ i\neq j}}^{n} x_j x_i\right) + E\left(\sum_{j}^{n} x_j^2\right)$$

$$= \left({}^{n}P_{2}\right) E(x_j x_i) + n\,E(x_j^2)$$

$$= \frac{n!}{(n-2)!}\left[(1)(p^2+q^2) + (-1)(2)(q)(p)\right]$$

$$+ n\left[(1)(p^2+q^2)\right]$$

$$= n(n-1)\left[p^2+q^2-2qp\right] + np^2 - nq^2$$

$$= n^2p^2 - 2n^2qp \cancel{- np^2} + 2nqp \cancel{+ np^2} - nq^2 + n^2q^2 - nq^2$$

$$= n^2p^2 - 2n^2(1-p)p + 2n(1-p)p - 2n(1-p)^2$$

$$+ n^2(1-p)^2.$$

$$= n^2p^2 - 2n^2p + 2n^2p^2 + 2np - 2np^2$$

$$- 2np^2 - 2n + 2np + n^2p^2 + n^2 - 2n^2p$$

$$= 4n^2p^2 - 4n^2p - 4np^2 + n^2 + 4np$$

$$(E(S_n))^2 = (n(2p-1))^2$$
$$= n^2(2p-1)^2$$
$$= n^2(4p^2-4p+1)$$
$$= 4n^2p^2 - 4n^2p + n^2$$

$$\therefore Var(S_n) = E(S_n^2) - (E(S_n))^2$$
$$= -4np^2 + 4np$$
$$= 4np(1-p)$$