SIMC: Introduction to Probability

Dillon Loh Guan Hui (06180194)

Problem 8 of Chapter 6 from Grimmett and Welsh's Text

Show that there exists a c such that the function

$$f(x,y) = \frac{c}{(1+x^2+y^2)^{3/2}}, \quad x,y \in \mathbb{R}$$

is a joint density function. Show that both marginal density functions of f are the density function of the Cauchy distribution.

By defn of density function,

$$1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x,y)$$

$$= c\int_{-\infty}^{\infty} dy \underbrace{\int_{-\infty}^{\infty} dx \frac{1}{(1+x^2+y^2)^{3/2}}}_{①}$$

①: First consider the indefinite integral

Sub $x = \sqrt{y^2+1}\tan(u)$

$$\Rightarrow dx = \sqrt{y^2+1}\sec^2(u)\, du$$

$$\Rightarrow \int dx \frac{1}{(1+(x^2+y^2))^{3/2}} = \int du \frac{\sqrt{y^2+1}\ \sec^2(u)}{[(y^2+1)\tan^2 u + y^2+1]^{3/2}}$$

$$= \int du \frac{\sqrt{y^2+1}\ \sec^2(u)}{(y^2+1)^{3/2}\sec^3(u)}$$

$$= \frac{1}{y^2+1}\int du\ \cos u$$

$$= \frac{1}{y^2+1}(\sin u) + C \quad , \quad C \in \mathbb{R}$$

$$= \frac{1}{y^2+1}\sin\left(\tan^{-1}\left(\frac{x}{\sqrt{y^2+1}}\right)\right) + C$$

since $\sin(\tan^{-1}(\theta)) = \frac{\theta}{\sqrt{\theta^2+1}}$

$$= \frac{1}{y^2+1}\frac{\frac{x}{\sqrt{y^2+1}}}{\sqrt{\frac{x^2}{y^2+1}+1}} + C$$

$$= \frac{x}{(y^2+1)\sqrt{x^2+y^2+1}} + C$$

Since the integrand is even with respect to x, the integral over symmetric interval $(-\infty, \infty)$ becomes

$$2\int_0^\infty dx \frac{1}{(1+x^2+y^2)^{3/2}}$$

$$\Rightarrow (1) = 2\left[\frac{x}{(y^2+1)\sqrt{x^2+y^2+1}}\right]_0^\infty$$

$$= 2\left[\lim_{A\to\infty}\left(\frac{1}{(y^2+1)\sqrt{1+\frac{y^2}{A^2}+\frac{1}{A^2}}}\right) - 0\right]$$

$$= 2\left(\frac{1}{y^2+1}\right)$$

$$= \frac{2}{y^2+1}$$

$$\Rightarrow 1 = c\int_{-\infty}^{\infty} dy \left(\frac{2}{y^2+1}\right)$$

$$= 2c\left[\tan^{-1}(y)\right]_{-\infty}^{\infty}$$

$$= 2c\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$= 2\pi c$$

$\therefore$ we can set $c = \frac{1}{2\pi}$ $\Rightarrow \exists c$

s.t. $f(x,y)$ is a joint density function

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{\frac{1}{2\pi}}{(1+x^2+y^2)^{3/2}}\, dy$$

? we solved this above.

$$= \frac{1}{2\pi} \frac{2}{x^2+1}$$

$$= \frac{1}{\pi} \frac{1}{x^2+1}$$ //

Similarly,

$$f_y(y) = \frac{1}{\pi} \frac{1}{y^2+1}$$ //

which are precisely the PDFs for the Standard Cauchy Distribution.

$\square$.