SMC : Introduction to Probability

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Problem 5.8.1 from Grimmett's and Welsh's Text.

The double exponential distribution (Laplace distribution) has pdf :

$$f(x) = \frac{1}{2} c e^{-c|x|}, \quad \text{for } x \in \mathbb{R}$$

where $c > 0$ is a parameter of the distribution. Show that the mean & variance of the distribution are 0 and $2c^{-2}$ respectively.

Mean :

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{2} c e^{-c|x|} \, dx$$

$$= \frac{c}{2} \int_{-\infty}^{\infty} x e^{-c|x|} \, dx$$

Let $f(x) = x e^{-c|x|}$

$$f(-x) = -x e^{-c|x|} = -f(x)$$

$\Rightarrow f$ is odd function

$\Rightarrow \mathbb{E}(x) = 0$ as integral of odd function over symmetric interval is 0.

$\therefore$ Mean $= 0$ $\Rightarrow$, X is integrable random variable

We check that this definition applys, by checking

$$\int_{-\infty}^{\infty} |x f(x)| \, dx = \frac{|c|}{2} \int_{-\infty}^{\infty} |x e^{-c|x|}| \, dx$$

$$= \frac{|c|}{2} \int_{-\infty}^{\infty} |x| \, |e^{-c|x|}| \, dx$$

Since $e^{-c|x|} > 0$ for any x

$$= \frac{|c|}{2} \int_{-\infty}^{\infty} \underbrace{|x| \, e^{-c|x|}}_{:= h(x) \Rightarrow h \text{ is even function}} \, dx$$

$$= (2) \frac{|c|}{2} \int_{0}^{\infty} x e^{-cx} \, dx$$

$$= |c| \left\{ \underbrace{\left[-\frac{e^{-cx}}{c} x \right]_0^{\infty}}_{= 0} + \left[\frac{e^{-cx}}{c} \right]_0^{\infty} \right\}$$

$$= -\frac{|c|}{c}$$, finite. if $c \neq 0$

Variance :-

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 \tfrac{1}{2} c e^{-c|x|} dx$$

$$= \frac{c}{2} \int_{-\infty}^{\infty} x^2 e^{-c|x|} dx$$

Let $g(x) = x^2 e^{-c|x|}$, clearly g is even functn

$$\Rightarrow E(X^2) = \frac{c}{2}(2) \int_0^{\infty} x^2 e^{-c|x|} dx$$

$|x| = x$ for $x \geq 0$

$$= \underbrace{c \int_0^{\infty} x^2 e^{-cx} dx}_{2}$$

$$2 = \underbrace{\left[-\frac{x^2 e^{-cx}}{c}\right]_0^{\infty}}_{=0} + \frac{2}{c}\int_0^{\infty} x e^{-cx} dx$$

$$= \frac{2}{c}\left\{\underbrace{\left[\frac{x e^{-cx}}{-c}\right]_0^{\infty}}_{=0} + \frac{1}{c}\int_0^{\infty} e^{-cx} dx\right\}$$

$$= \frac{2}{c}\left[-\frac{1}{c^2}e^{-cx}\right]_0^{\infty}$$

$$= 0 + \frac{2}{c^3} = \frac{2}{c^3}$$

$$\Rightarrow E(X^2) = c\,\frac{2}{c^3} = \frac{2}{c^2}$$

$$\therefore Var(X^2) = E(X^2) - (E(X))^2$$

$$= \frac{2}{c^2} - 0$$

$$= 2c^{-2} \quad \text{h}$$

$$\square.$$

Again, to check if defn of $E(X^2)$ applys, check

$$\int_{-\infty}^{\infty} |x^2 f(x)|\,dx = \left|\frac{|c|}{2}\int_{-\infty}^{\infty} x^2 e^{-c|x|}\,dx\right|$$

$$= \frac{|c|}{2}\int_{-\infty}^{\infty} x^2 e^{-c|x|}\,dx$$

which clearly is defined.