

# SMC: Introduction to Probability

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Problem 5.8.1 from Grimmett's and Welsh's Text.

The double exponential distribution (Laplace distribution) has pdf:

$$f(x) = \frac{1}{2} c e^{-c|x|}, \text{ for } x \in \mathbb{R}$$

where  $c > 0$  is a parameter of the distribution. Show that the mean & variance of the distribution are 0 and  $2c^{-2}$  respectively.

Mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{2} c e^{-c|x|} dx$$

$$= \frac{c}{2} \int_{-\infty}^{\infty} x e^{-c|x|} dx$$

$$\text{Let } f(x) = x e^{-c|x|}$$

$$f(-x) = -x e^{-c|x|} = -f(x)$$

$\Rightarrow f$  is odd function

$\Rightarrow E(X) = 0$  as integral of odd function over symmetric interval is 0.

$\therefore \text{Mean} = 0$   $\hookrightarrow$  ,  $X$  is integrable random variable

We check that this definition applies, by checking

$$\begin{aligned} \int_{-\infty}^{\infty} |x f(x)| dx &= \frac{|c|}{2} \int_{-\infty}^{\infty} |x e^{-c|x|}| dx \\ &= \frac{|c|}{2} \int_{-\infty}^{\infty} |x| e^{-c|x|} dx \end{aligned}$$

Since  $e^{-c|x|} > 0$  for any  $x$

$$\begin{aligned} &= \frac{|c|}{2} \int_{-\infty}^{\infty} |x| e^{-c|x|} dx \\ &\quad \underbrace{\hspace{10em}}_{:= h(x) \Rightarrow h \text{ is even function}} \end{aligned}$$

$$= \frac{|c|}{2} \int_0^{\infty} x e^{-cx} dx$$

$$= |c| \left\{ \underbrace{\left[ -\frac{e^{-cx}}{c} x \right]_0^{\infty}}_{=0} + \left[ \frac{e^{-cx}}{c} \right]_0^{\infty} \right\}$$

$$= -\frac{|c|}{c}, \text{ finite, if } c \neq 0$$

Variance:

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_{-\infty}^{\infty} x^2 \frac{1}{2} c e^{-c|x|} dx \\ &= \frac{c}{2} \int_{-\infty}^{\infty} x^2 e^{-c|x|} dx \end{aligned}$$

Let  $g(x) = x^2 e^{-c|x|}$ , clearly  $g$  is even.  $\therefore$

$$\begin{aligned} \Rightarrow E(X^2) &= \frac{c}{2} \int_0^{\infty} x^2 e^{-c|x|} dx \\ &= c \int_0^{\infty} x^2 e^{-cx} dx \end{aligned}$$

$\swarrow$   $|x| = x$   
for  $x \geq 0$

$$\textcircled{2} = \underbrace{\left[ -\frac{x^2 e^{-cx}}{c} \right]_0^{\infty}}_{=0} + \frac{2}{c} \int_0^{\infty} x e^{-cx} dx$$

$$= \frac{2}{c} \left\{ \underbrace{\left[ \frac{x e^{-cx}}{-c} \right]_0^{\infty}}_{=0} + \frac{1}{c} \int_0^{\infty} e^{-cx} dx \right\}$$

$$= \frac{2}{c} \left[ -\frac{1}{c^2} e^{-cx} \right]_0^{\infty}$$

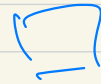
$$= 0 + \frac{2}{c^3} = \frac{2}{c^3}$$

$$\Rightarrow E(X^2) = c \frac{2}{c^3} = \frac{2}{c^2}$$

$$\therefore \text{Var}(X^2) = E(X^2) - (E(X))^2$$

$$= \frac{2}{c^2} - 0$$

$$= 2c^{-2} \quad \checkmark$$



Again, to check if defn of  $E(X^2)$  applies, check

$$\int_{-\infty}^{\infty} |x^2 f(x)| dx = \frac{1}{2} \int_{-\infty}^{\infty} |x^2 e^{-c|x|} dx|$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-c|x|} dx$$

which clearly is defined.