Expectations of Continuous random variables.

Exercise 5.68.

The random variable X has density function

$$f(x) = cx(1-x) \quad \text{for} \quad 0 \le x \le 1.$$

- Determine c.

$$\mathbb{F}(x) = \int_0^1 f(x)\,dx = \int_0^1 |\,cx(1-x)\,|\,dx = c\int_0^1 (x - x^2)\,dx =$$

$$= c\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = c\left(\frac{1}{2} - \frac{1}{3}\right) = c\cdot\frac{1}{6} = 1.$$

$$\underline{\underline{c = 6}}$$

- Find the mean and variance of X

the mean value of X

$$\mathbb{E}(X) = \int_0^1 x\, f_X(x)\,dx = \int_0^1 x\cdot cx(1-x) = 6\int_0^1 (x^2 - x^3)\,dx =$$

$$= 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \underline{\underline{\frac{1}{2}}}$$

$$var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

$$\mathbb{E}(X^2) = \int_0^1 x^2 cx(1-x)\,dx = 6\int_0^1 (x^3 - x^4)\,dx = 6\cdot\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 =$$

$$= 6\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{6}{20} = \frac{3}{10}$$

$$var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \underline{\underline{\frac{1}{20}}}$$

5.8.7. If X is a continuous random variable taking non-negative values only, show that

$$\mathbb{E}(X) = \int_0^\infty (1 - F_X(x))\,dx \quad \text{whenever this integral exists}$$

$$\mathbb{E}(X) = \int_0^\infty (1 - F_X(x))\,dx = \int_0^\infty \left(1 - \int_0^x f_X(s)\,ds\right) dx =$$

Note that $F_X(x) = \int_0^\infty f_X(s)\,ds$ goes to 1 as $x \to \infty$.

$$= \int_0^\infty \left(1 - \int_0^x f_X(s)\,ds\right) dx = \int_0^\infty \int_x^\infty (f_X(s)\,ds)\,dx =$$

$$= \int_0^\infty \left(\int_0^s f_X(s)\,dx\right) ds = \int_0^\infty f_X(s) \cdot s\,ds = \int_0^\infty s\,f_X(s)\,ds = E(X) \quad \square$$

5.8.8. Show that $E(g(X)) = \int_{-\infty}^\infty g(x)\,f_X(x)\,dx$

let's set $Y = g(x)$

Using the result from 5.8.7. we can write.

$$\mathbb{E}(Y) = \int_0^\infty (1 - F_Y(y))\,dy = \int_0^\infty \left(1 - \int_0^\infty f_Y(y)\,dy\right) dy =$$

$$= \underbrace{y(1 - F_Y(y))\Big|_0^\infty}_{0 \text{ because } \lim_{y\to\infty} F_Y(y) = 1} + \int_0^\infty y\,f_Y(y)\,dy =$$

by theorem 5.50 $f_Y(y) = f_X(g^{-1}(y))\,\frac{d}{dy}[g^{-1}(y)]$

$$\int_0^\infty y \cdot f_Y(y)\,dy = \int_0^\infty y \cdot f_X(g^{-1}(y))\,\underbrace{\frac{d}{dy}(g^{-1}(y)) \cdot dy}_{d(g^{-1}(y))} =$$

$$= \int_0^\infty y\,f_X(g^{-1}(y))\,d(g^{-1}(y)) = \int_0^\infty g(x)\,f_X(x)\,dx \quad \square$$