Graphs and Epidemiology

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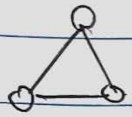
Exercise 1 (Lecture note p.20)

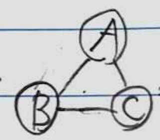
Find a basis for the example of 3 nodes such that the transition matrix in block diagonal.
Write this matrix explicitly.

Recall that the model of the example is the following.



- A complete graph with 3 nodes, and 2 states a state S and I.
  A rate that each node change state from I to S is γ, and that each node has state S in a pair S–I change state to I is J.

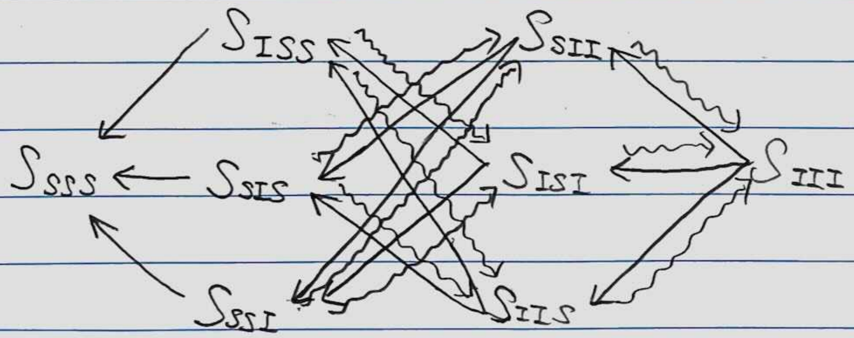
Set X_{ABC} is a probability that system in state



The model's state space is

$$S = \{ S_{SSS}, S_{ISS}, S_{SIS}, S_{SSI}, S_{SII}, S_{ISI}, S_{IIS}, S_{III} \}$$

and a diagram of transition is as following;



$\longrightarrow$: $I \to S$

$\rightsquigarrow$: $S \to I$

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So we obtain a master equation :

$$
\begin{cases}
\dot{X}_{SSS} = \gamma(X_{ISS} + X_{SIS} + X_{SSI}) \\
\dot{X}_{ISS} = \gamma(X_{ISI} + X_{IIS}) - (\gamma+2J)X_{ISS} \\
\dot{X}_{SIS} = \gamma(X_{IIS} + X_{SII}) - (\gamma+2J)X_{SIS} \\
\dot{X}_{SSI} = \gamma(X_{ISI} + X_{SII}) - (\gamma+2J)X_{SSI} \\
\dot{X}_{SII} = J(X_{SIS} + X_{SSI}) + \gamma X_{III} - (2\gamma+2J)X_{SII} \\
\dot{X}_{ISI} = J(X_{ISS} + X_{SSI}) + \gamma X_{III} - (2\gamma+2J)X_{ISI} \\
\dot{X}_{IIS} = J(X_{ISS} + X_{SIS}) + \gamma X_{III} - (2\gamma+2J)X_{IIS} \\
\dot{X}_{III} = 2J(X_{SII} + X_{ISI} + X_{IIS}) - 3\gamma X_{III}
\end{cases}
$$

Then set $X = {}^t(X_{SSS}\ X_{ISS}\ X_{SIS}\ X_{SSI}\ X_{SII}\ X_{ISI}\ X_{IIS}\ X_{III})$,
the master equation is written as $\dot{X} = TX$
where

$$
T = \left(\begin{array}{c|ccc|ccc|c}
0 & \gamma & \gamma & \gamma & 0 & 0 & 0 & 0 \\
\hline
0 & -(\gamma+2J) & 0 & 0 & 0 & \gamma & \gamma & 0 \\
0 & 0 & -(\gamma+2J) & 0 & \gamma & 0 & \gamma & 0 \\
0 & 0 & 0 & -(\gamma+2J) & \gamma & \gamma & 0 & 0 \\
\hline
0 & 0 & J & J & -(2\gamma+2J) & 0 & 0 & \gamma \\
0 & J & 0 & J & 0 & -(2\gamma+2J) & 0 & \gamma \\
0 & J & J & 0 & 0 & 0 & -(2\gamma+2J) & \gamma \\
\hline
0 & 0 & 0 & 0 & 2J & 2J & 2J & -3\gamma
\end{array}\right)
$$

So where $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$,
set matrices as $B_0 = (0)$, $C_0 = (\gamma\ \gamma\ \gamma)$,
$A_1 = {}^t(0\ 0\ 0)$, $B_1 = -(\gamma+2J)I$, $C_1 = \gamma K$,
$A_2 = JK$, $B_2 = -(2\gamma+2J)I$, $C_1 = {}^t(\gamma\ \gamma\ \gamma)$
$A_3 = (2J\ 2J\ 2J)$, $B_3 = (-3\gamma)$, for block diagonal matrix

$$
T = \begin{pmatrix} B_0 & C_0 & 0 & 0 \\ A_1 & B_1 & C_1 & 0 \\ 0 & A_2 & B_2 & C_2 \\ 0 & 0 & A_3 & B_3 \end{pmatrix}
$$

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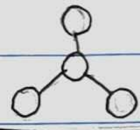
Exercise 2 (Lecture note p.26, [KMS] p.37, 38)

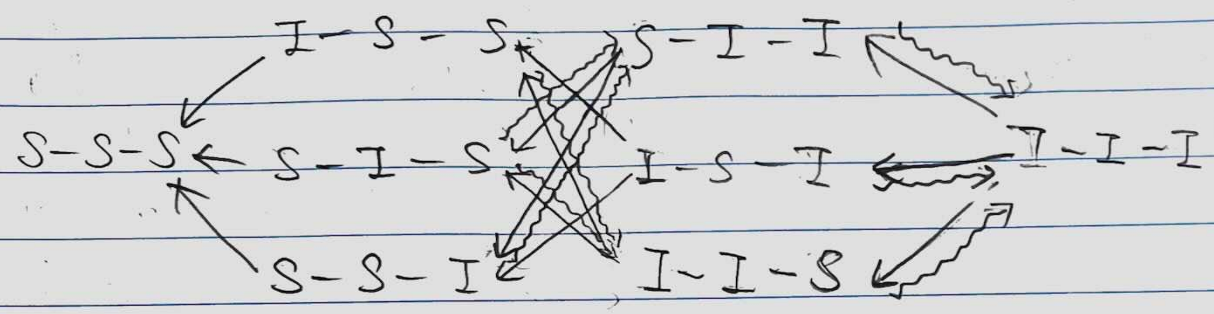
Find the matrices for the SIS dynamics in the following graphs and write down the full system of master equations

(a) 

(b) 

(c) 

(a) the transition diagram is



then, the master equation is

$$\dot{X}_{SSS} = \gamma(X_{ISS}+X_{SIS}+X_{SSI})$$
$$\dot{X}_{ISS} = \gamma(X_{ISI}+X_{IIS}) - (\gamma+J)X_{ISS}$$
$$\dot{X}_{SIS} = \gamma(X_{SII}+X_{IIS}) - (\gamma+2J)X_{SIS}$$
$$\dot{X}_{SSI} = \gamma(X_{SII}+X_{ISI}) - (\gamma+J)X_{SSI}$$
$$\dot{X}_{SII} = J(X_{SIS}+X_{SSI}) + \gamma X_{III} - (2\gamma+J)X_{SII}$$
$$\dot{X}_{ISI} = \gamma X_{III} - (2\gamma+2J)X_{ISI}$$
$$\dot{X}_{IIS} = J(X_{ISS}+X_{SIS}) + \gamma X_{III} - (2\gamma+J)X_{IIS}$$
$$\dot{X}_{III} = J(X_{SII}+2X_{ISI}+X_{IIS}) - 3\gamma X_{III}$$

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So the matrix is following for the same X as Exercise 1

$$\left(\begin{array}{c|ccc|ccc|c}
0 & \gamma & \gamma & \gamma & & & & \\ \hline
 & -(\gamma+J) & & & & \gamma & \gamma & \\
 & & -(\gamma+2J) & & \gamma & & \gamma & \\
 & & & -(\gamma+J) & \gamma & \gamma & & \\ \hline
 & & J & J & -(2\gamma+J) & & & \gamma \\
 & & & & & -(2\gamma+2J) & & \gamma \\
 & J & J & & & & -(2\gamma+J) & \gamma \\ \hline
 & & & & & & & -3\gamma
\end{array}\right)$$

(where the blank means 0)

and set $B^0 = (0)$, $C^0 = (\gamma\ \gamma\ \gamma)$

$$A^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\quad B^1 = \begin{pmatrix} -(\gamma+J) & & \\ & -(\gamma+J) & \\ & & -(\gamma+J) \end{pmatrix},\quad C^1 = \gamma K$$

$$A^2 = \begin{pmatrix} 0 & J & J \\ 0 & 0 & 0 \\ J & J & 0 \end{pmatrix},\quad B^2 = \begin{pmatrix} -(2\gamma+J) & & \\ & -(2\gamma+2J) & \\ & & -(2\gamma+J) \end{pmatrix},\quad C^2 = \begin{pmatrix} \gamma \\ \gamma \\ \gamma \end{pmatrix}$$

$$A^3 = (0\ 0\ 0),\quad B^3 = (-3\gamma)$$

as the matrix can be written as

$$\begin{pmatrix} B^0 & C^0 & 0 & 0 \\ A^1 & B^1 & C^1 & 0 \\ 0 & A^2 & B^2 & C^2 \\ 0 & 0 & A^3 & B^3 \end{pmatrix}$$

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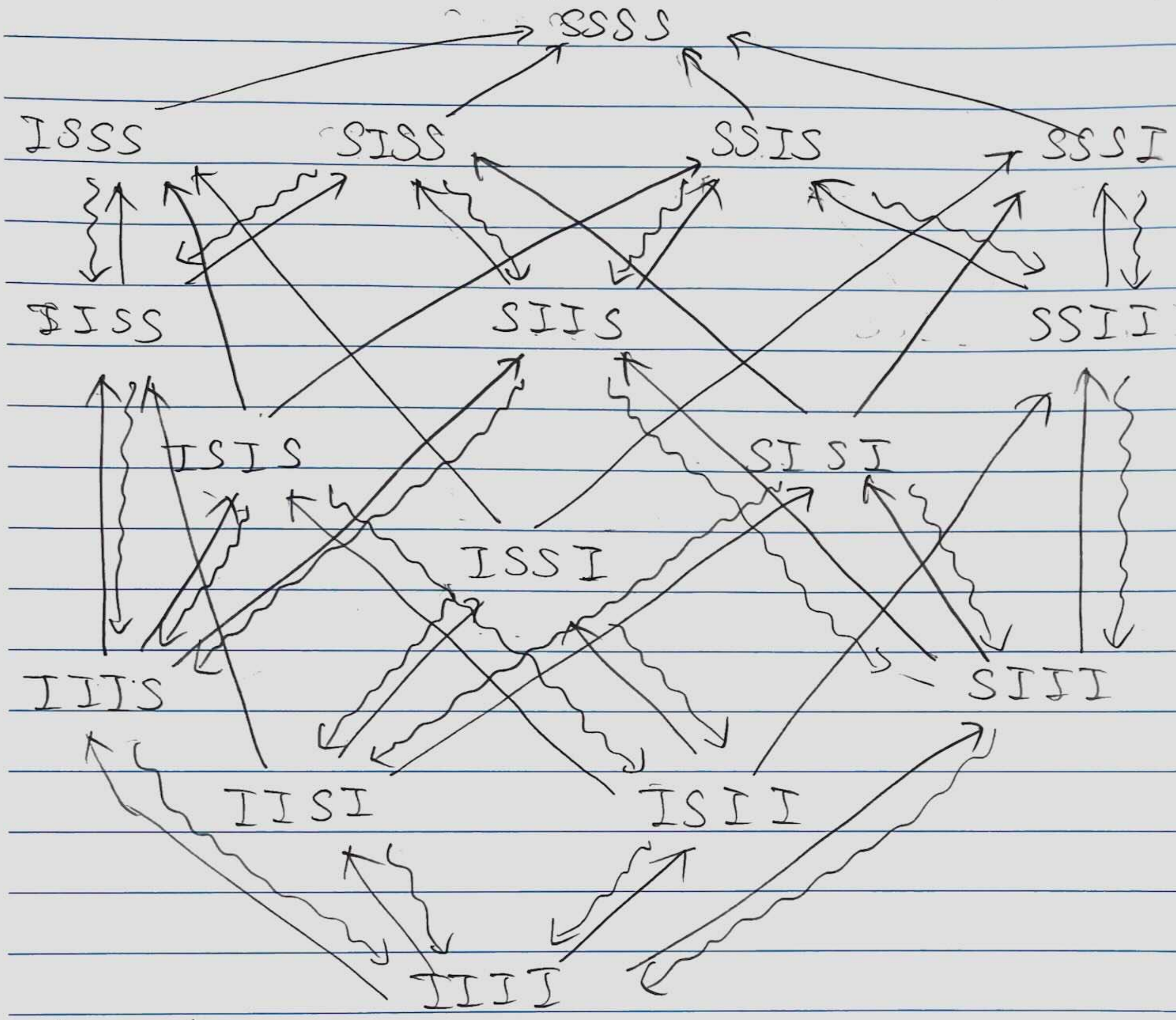
(b) the transition diagram is



then the master equation is

$$\dot{X}_{SSSS} = \gamma(X_{ISSS} + X_{SISS} + X_{SSIS} + X_{SSSI})$$

$$\dot{X}_{ISSS} = \gamma(X_{IISS} + X_{ISIS} + X_{ISSI}) - (\gamma + J)X_{ISSS}$$

$$\dot{X}_{SISS} = \gamma(X_{IISS} + X_{SIIS} + X_{SISI}) - (\gamma + 2J)X_{SISS}$$

$$\dot{X}_{SSIS} = \gamma(X_{ISIS} + X_{SIIS} + X_{SSII}) - (\gamma + 2J)X_{SSIS}$$

$$\dot{X}_{SSSI} = \gamma(X_{ISSI} + X_{SISI} + X_{SSII}) - (\gamma + J)X_{SSSI}$$

$$\dot{X}_{IISS} = \gamma(X_{IIIS} + X_{IISI}) + J(X_{ISSS} + X_{SISS}) - (2\gamma + J)X_{IISS}$$

$$\dot{X}_{ISIS} = \gamma(X_{IIIS} + X_{ISII}) - (2\gamma + 3J)X_{ISIS}$$

$$\dot{X}_{SIIS} = \gamma(X_{IIIS} + X_{SIII}) + J(X_{SISS} + X_{SSIS}) - (2\gamma + 2J)X_{SIIS}$$

$$\dot{X}_{ISSI} = \gamma(X_{IISI} + X_{ISII}) - (2\gamma + 2J)X_{ISSI}$$

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$\dot{X}_{SISI} = \gamma(X_{IISI} + X_{SIII}) - (2\gamma + 3J) X_{SISI}$

$\dot{X}_{SSII} = \gamma(X_{ISII} + X_{SIII}) + J(X_{SSIS} + X_{SSSI}) - (2\gamma + J) X_{SSII}$

$\dot{X}_{IIIS} = \gamma X_{IIII} + J(X_{IISS} + 2X_{ISIS} + X_{SIIS}) - (3\gamma + J) X_{IIIS}$

$\dot{X}_{IISI} = \gamma X_{IIII} + J(X_{ISSI} + X_{SISI}) - (3\gamma + 2J) X_{IISI}$

$\dot{X}_{ISII} = \gamma X_{IIII} + J(X_{ISSI} + X_{ISIS}) - (3\gamma + 2J) X_{ISII}$

$\dot{X}_{SIII} = \gamma X_{IIII} + J(X_{SSII} + 2X_{SISI} + X_{SIIS}) - (3\gamma + J) X_{SIII}$

$\dot{X}_{IIII} = J(X_{IIIS} + 2X_{IISI} + 2X_{ISII} + X_{SIII}) - 4\gamma X_{IIII}$

So the matrix for $X = (X_{SSSS}, X_{ISSS}, X_{SISS}, X_{SSIS}, X_{SSSI},$
$X_{IISS}, X_{ISIS}, X_{SIIS}, X_{ISSI}, X_{SISI}, X_{SSII},$
$X_{IIIS}, X_{IISI}, X_{ISII}, X_{SIII}, X_{IIII})$

is written as

$$\begin{pmatrix} B^0 & C^0 & 0 & 0 & 0 \\ A^1 & B^1 & C^1 & 0 & 0 \\ 0 & A^2 & B^2 & C^2 & 0 \\ 0 & 0 & A^3 & B^3 & C^3 \\ 0 & 0 & 0 & A^4 & B^4 \end{pmatrix}$$

where

$B^0 = (0)$, $C^0 = (\gamma\ \gamma\ \gamma\ \gamma)$.

$$A^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad B^1 = \begin{pmatrix} -(\gamma+J) & 0 & 0 & 0 \\ 0 & -(\gamma+2J) & 0 & 0 \\ 0 & 0 & -(\gamma+2J) & 0 \\ 0 & 0 & 0 & -(\gamma+J) \end{pmatrix},$$

$$C^1 = \begin{pmatrix} \gamma & \gamma & 0 & \gamma & 0 & 0 \\ \gamma & 0 & \gamma & 0 & \gamma & 0 \\ 0 & \gamma & \gamma & 0 & 0 & \gamma \\ 0 & 0 & 0 & \gamma & \gamma & \gamma \end{pmatrix},$$

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$$A^2 = \begin{pmatrix} J & J & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & J & J & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & J & J \end{pmatrix},$$

$$B^2 = \begin{pmatrix} -(2\gamma+J) & 0 & 0 & 0 & 0 & 0 \\ 0 & -(2\gamma+3J) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(2\gamma+2J) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(2\gamma+2J) & 0 & 0 \\ 0 & 0 & 0 & 0 & -(2\gamma+3J) & 0 \\ 0 & 0 & 0 & 0 & 0 & -(2\gamma+J) \end{pmatrix}$$

$$C^2 = \begin{pmatrix} \gamma & \gamma & 0 & 0 \\ \gamma & 0 & \gamma & 0 \\ \gamma & 0 & 0 & \gamma \\ 0 & \gamma & \gamma & 0 \\ 0 & \gamma & 0 & \gamma \\ 0 & 0 & \gamma & \gamma \end{pmatrix}, \quad A^3 = \begin{pmatrix} J & 2J & J & 0 & 0 & 0 \\ 0 & 0 & 0 & J & J & 0 \\ 0 & J & 0 & J & 0 & 0 \\ 0 & 0 & J & 0 & 2J & J \end{pmatrix}$$

$$B^3 = \begin{pmatrix} -(3\gamma+J) & 0 & 0 & 0 \\ 0 & -(3\gamma+2J) & 0 & 0 \\ 0 & 0 & -(3\gamma+2J) & 0 \\ 0 & 0 & 0 & -(3\gamma+J) \end{pmatrix}, \quad C^3 = \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix},$$

$$A^4 = \begin{pmatrix} J & 2J & 2J & J \end{pmatrix}, \quad B^4 = \begin{pmatrix} -4\gamma \end{pmatrix} \quad //$$

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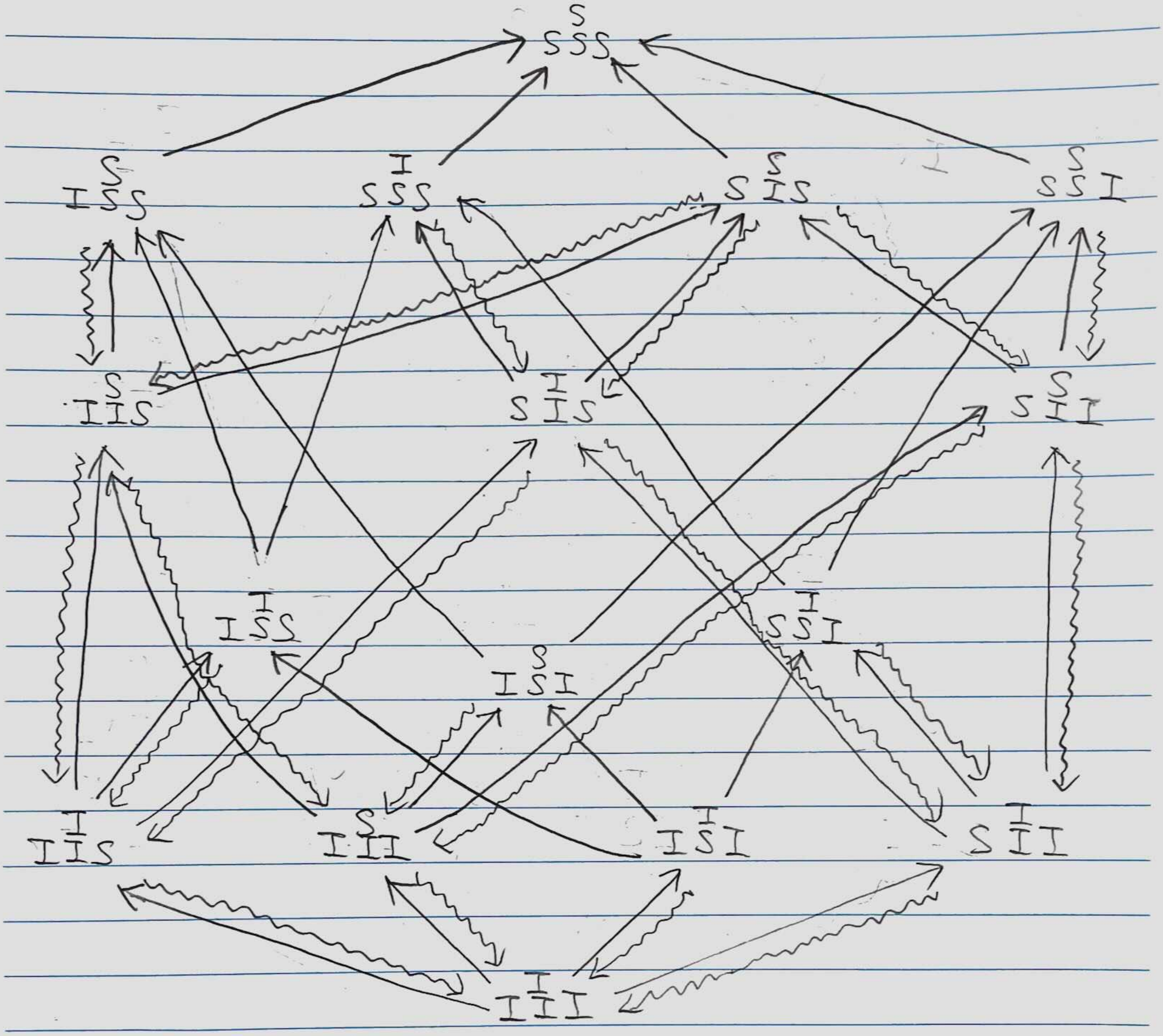
(c) the transition diagram is



then, the master equation is

$$\dot{X}_{S\overset{S}{S}S} = \gamma\left(X_{I\overset{S}{S}S} + X_{S\overset{I}{S}S} + X_{S\overset{S}{I}S} + X_{S\overset{S}{S}I}\right)$$

$$\dot{X}_{I\overset{S}{S}S} = \gamma\left(X_{I\overset{S}{I}S} + X_{I\overset{I}{S}S} + X_{I\overset{S}{S}I}\right) - (\gamma + J)\,X_{I\overset{S}{S}S}$$

$$\dot{X}_{S\overset{I}{S}S} = \gamma\left(X_{S\overset{I}{S}I} + X_{S\overset{I}{I}S} + X_{I\overset{I}{S}S}\right) - (\gamma + J)\,X_{S\overset{I}{S}S}$$

$$\dot{X}_{S\overset{S}{I}S} = \gamma\left(X_{I\overset{S}{I}S} + X_{S\overset{I}{I}S} + X_{S\overset{S}{I}I}\right) - (\gamma + 3J)\,X_{S\overset{S}{I}S}$$

$$\dot{X}_{S\overset{S}{S}I} = \gamma\left(X_{S\overset{I}{S}I} + X_{I\overset{S}{S}I} + X_{S\overset{S}{I}I}\right) - (\gamma + J)\,X_{S\overset{S}{S}I}$$

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$$\dot{X}_{I\overset{S}{I}S} = J(X_{I\overset{S}{S}S} + X_{S\overset{S}{I}S}) + \gamma(X_{I\overset{S}{I}I} + X_{I\overset{I}{I}S}) - (2\gamma+2J)X_{I\overset{S}{I}S}$$

$$\dot{X}_{S\overset{I}{I}S} = J(X_{S\overset{I}{S}S} + X_{S\overset{S}{I}S}) + \gamma(X_{S\overset{I}{I}I} + X_{I\overset{I}{I}S}) - (2\gamma+2J)X_{S\overset{I}{I}S}$$

$$\dot{X}_{S\overset{S}{I}I} = J(X_{S\overset{S}{I}S} + X_{S\overset{S}{S}I}) + \gamma(X_{S\overset{I}{I}I} + X_{I\overset{S}{I}I}) - (2\gamma+2J)X_{S\overset{S}{I}I}$$

$$\dot{X}_{S\overset{I}{S}I} = \gamma(X_{S\overset{I}{I}I} + X_{I\overset{I}{S}I}) - (2\gamma+2J)X_{S\overset{I}{S}I}$$

$$\dot{X}_{I\overset{S}{S}I} = \gamma(X_{I\overset{S}{I}I} + X_{I\overset{I}{S}I}) - (2\gamma+2J)X_{I\overset{S}{I}I}$$

$$\dot{X}_{I\overset{I}{S}S} = \gamma(X_{I\overset{I}{S}I} + X_{I\overset{I}{I}S}) - (2\gamma+2J)X_{I\overset{I}{S}S}$$

$$\dot{X}_{S\overset{I}{I}I} = J(2X_{S\overset{I}{S}I} + X_{S\overset{I}{I}S} + X_{S\overset{S}{I}I}) + \gamma X_{I\overset{I}{I}I} - (3\gamma+J)X_{S\overset{I}{I}I}$$

$$\dot{X}_{I\overset{S}{I}I} = J(X_{I\overset{S}{I}S} + 2X_{I\overset{S}{S}I} + X_{S\overset{S}{I}I}) + \gamma X_{I\overset{I}{I}I} - (3\gamma+J)X_{I\overset{S}{I}I}$$

$$\dot{X}_{I\overset{I}{S}I} = \gamma X_{I\overset{I}{I}I} - (3\gamma+3J)X_{I\overset{I}{S}I}$$

$$\dot{X}_{I\overset{I}{I}S} = J(X_{I\overset{S}{I}S} + X_{S\overset{I}{I}S} + 2X_{I\overset{I}{S}S}) + \gamma X_{I\overset{I}{I}I} - (3\gamma+J)X_{I\overset{I}{I}S}$$

$$\dot{X}_{I\overset{I}{I}I} = J(X_{S\overset{I}{I}I} + X_{I\overset{S}{I}I} + 3X_{I\overset{I}{S}I} + X_{I\overset{I}{I}S}) - 4\gamma X_{I\overset{I}{I}I}$$

So the diagonal matrix for

$$X = (X_{S\overset{S}{S}S}, X_{I\overset{S}{S}S}, X_{S\overset{I}{S}S}, X_{S\overset{S}{I}S}, X_{S\overset{S}{S}I}, X_{I\overset{S}{I}S}, X_{S\overset{I}{I}S}, X_{S\overset{S}{I}I}, X_{S\overset{I}{S}I}, X_{I\overset{S}{S}I}, X_{I\overset{I}{S}S}, X_{S\overset{I}{I}I}, X_{I\overset{S}{I}I}, X_{I\overset{I}{S}I}, X_{I\overset{I}{I}S}, X_{I\overset{I}{I}I})$$

is written as

$$\begin{pmatrix} B^0 & C^0 & 0 & 0 & 0 \\ A^1 & B^1 & C^1 & 0 & 0 \\ 0 & A^2 & B^2 & C^2 & 0 \\ 0 & 0 & A^3 & B^3 & C^3 \\ 0 & 0 & 0 & A^4 & B^4 \end{pmatrix}$$

where

$$B^0 = (0), \quad C^0 = (\gamma \ \gamma \ \gamma \ \gamma),$$

$$A^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad B^1 = \begin{pmatrix} -(\gamma+J) & 0 & 0 & 0 \\ 0 & -(\gamma+J) & 0 & 0 \\ 0 & 0 & -(\gamma+3J) & 0 \\ 0 & 0 & 0 & -(\gamma+J) \end{pmatrix},$$

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$$C^1 = \begin{pmatrix} \gamma & 0 & 0 & 0 & \gamma & \gamma \\ 0 & \gamma & 0 & \gamma & 0 & \gamma \\ \gamma & \gamma & \gamma & 0 & 0 & 0 \\ 0 & 0 & \gamma & \gamma & \gamma & 0 \end{pmatrix},$$

$$A^2 = \begin{pmatrix} J & 0 & J & 0 \\ 0 & J & J & 0 \\ 0 & 0 & J & J \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B^2 = \begin{pmatrix} -(2\gamma+2J) & 0 & 0 & 0 & 0 & 0 \\ 0 & -(2\gamma+2J) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(2\gamma+2J) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(2\gamma+2J) & 0 & 0 \\ 0 & 0 & 0 & 0 & -(2\gamma+2J) & 0 \\ 0 & 0 & 0 & 0 & 0 & -(2\gamma+2J) \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 0 & \gamma & 0 & \gamma \\ \gamma & 0 & 0 & \gamma \\ \gamma & \gamma & 0 & 0 \\ \gamma & 0 & \gamma & 0 \\ 0 & \gamma & \gamma & 0 \\ 0 & 0 & \gamma & \gamma \end{pmatrix}, A^3 = \begin{pmatrix} 0 & J & J & 2J & 0 & 0 \\ J & 0 & J & 0 & 2J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ J & J & 0 & 0 & 0 & 2J \end{pmatrix},$$

$$B^3 = \begin{pmatrix} -(3\gamma+J) & 0 & 0 & 0 \\ 0 & -(3\gamma+J) & 0 & 0 \\ 0 & 0 & -(3\gamma+3J) & 0 \\ 0 & 0 & 0 & -(3\gamma+J) \end{pmatrix}, C^3 = \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix},$$

$$A^4 = \begin{pmatrix} J & J & 3J & J \end{pmatrix}, \quad B^4 = \begin{pmatrix} -4\gamma \end{pmatrix} \qquad // \square$$