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Report 2:

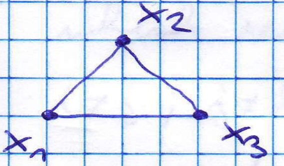
Let $G=(V,E)$ be a finite undirected graph.

Let $V=\{x_1,x_2,\dots,x_n\}$ the set of vertices and $E \subseteq V\times V$ the set of edges. Let $A\in M_{n\times n}(\mathbb{R})$ the adjacency matrix, which satisfies $a_{ij}=1$ if $(x_i,x_j)\in E$ and $a_{ij}=0$ if $(x_i,x_j)\notin E$. We don't allow loops, so $\forall\, 1\le i\le n:\ (x_i,x_i)\notin E \Rightarrow \forall\, 1\le i\le n:\ A_{ii}=a_{ii}=0$.

We know that for an undirected graph, we have $(x_i,x_j)\in E \Longrightarrow (x_j,x_i)\in E$ and therefore $A$ is symmetric.

Let $tr(A):=\sum_{i=1}^{n} a_{ii}$ be the trace of a matrix and $\|A\|_1=\sum_{i,j=1}^{n} a_{ij}$ the sum of all values of a matrix.

First we want to show a counterexample, that the formula from the book [KMS, page 127] is wrong!




$$\Rightarrow A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix} \Rightarrow A^2=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix} \Rightarrow A^3=\begin{pmatrix}2&3&3\\3&2&3\\3&3&2\end{pmatrix}$$

The formula says $\dfrac{\text{number of } \cancel{\text{triangles}} \text{ triangles}}{\text{number of triples}} = \dfrac{tr(A^3)}{\|A^2\|-tr(A^2)}$

In this example we have $tr(A^3)=6$, $\|A^2\|=12$, $tr(A^2)=6$

$\Longrightarrow \dfrac{6}{12-6}=1$, but there are 1 triangle $(x_1,x_2,x_3)$ and 3 triples $(x_1,x_2,x_3)$; $(x_1,x_3,x_2)$; $(x_2,x_1,x_3)$.

So the formula should give us $\frac{1}{3}$ as the solution.

The right formula, which we will proof, is

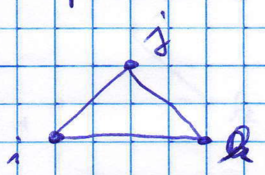
$$\frac{\text{number of triangles}}{\text{number of triples}} = \frac{1}{3}\,\frac{tr(A^3)}{\|A^2\|-tr(A^2)}$$

Remark: A triple is given by a middle point and a set of 2 different points, where this 2 points have an edge to the middle point, so $(x_1,x_2,x_3)=(x_3,x_2,x_1)$ it is the same triple.

Proof: The important fact, what we need, is the Theorem from the lecture page 7.

Theo: Let $A$ be the adjacency matrix of graph $G$ with the same assumptions as for page 1 of this report. Then $\forall r \geq 1$: $(A^r)_{i,j} \in \mathbb{R}$ is the number of different walks from $i$ to $j$ of length $r$.

First we calculate the number of triangles:
From the Theo. we know, that from $A^3$ we get the number of walks of length 3. A triangle is a walk of length 3 with the same starting and ending point. Therefore we get with $tr(A^3) = \sum_{i=1}^{n} (A^3)_{i,i}$ the sum of all walks, which is a triangle. But there are different walks, which represent the same triangle. Let $i, j, k \in V$ and $i \neq j$ ; $j \neq k$ ; $k \neq i$ then there are exactly 6 walks, which represent the same triangle.




| | |
|---|---|
| $i \to j \to k \to i$ | $i \to k \to j \to i$ |
| $j \to k \to i \to j$ | $j \to i \to k \to j$ |
| $k \to i \to j \to k$ | $k \to j \to i \to k$ |

We have 3 different ~~ed~~ vertices, where the walk can start, and for each of this walks, we have a walk in the other direction (one time clockwise direction and one time anticlockwise direction).
So we conclude that "number of triangles" $= \frac{tr(A^3)}{6}$

Second we calculate the number of triples:

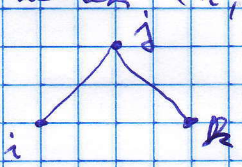
From the Theo. we know, that from $A^2$ we get the number of walks of length 2. A triple is ~~so~~ a walk of length 2, where the starting and ending point is different.

Therefore we get with $\|A^2\| - tr(A^2) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (A^2)_{ij}$ the sum of all walks, which is a triple. But there are different walks, which represent the same triple.

Set $i, j, k \in V$ and $i \neq j$, $j \neq k$, $i \neq k$ then there are exactly 2 walks, which represent the same triple.

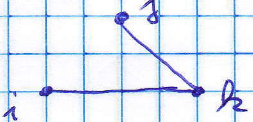
When $(i, j, k)$ have 0 or 1 edge, then there is no walk of length 2 for this 3 vertices.

When $(i, j, k)$ have 2 edges, then there are 2 walks of length 2, which represent the same triple.



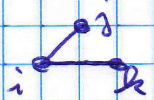

$i \to j \to k$ and $k \to j \to i$

For every possibility of 2 edges, we have 2 walks

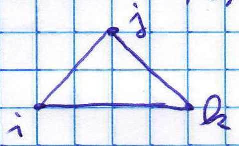



$i \to k \to j$ and $j \to k \to i$

analog




When $(i, j, k)$ have 3 edges, then there are 6 walks of length 2, which represent 3 different triples




$i \to j \to k$ and $k \to j \to i$

$j \to i \to k$ and $k \to i \to j$

$j \to k \to i$ and $i \to k \to j$

So we count in every case every triple 2 times.

$\Rightarrow$ "number of triples" $= \frac{1}{2} \left[ \|A^2\| - tr(A^2) \right]$

Together we get

$$\frac{\text{number of triangles}}{\text{number of triples}} = \frac{\frac{tr(A^3)}{6}}{\frac{\|A^2\| - tr(A^2)}{2}} = \frac{1}{3} \frac{tr(A^3)}{\|A^2\| - tr(A^2)}$$