Homework 6

Exercise 1 *Let us consider* $f:\mathbb{R}^3\to\mathbb{R}$ *and* $F:\mathbb{R}^3\to\mathbb{R}^3$ *sufficiently differentiable on* $\mathbb{R}^3$*. One also sets* $F={}^t(F_1,F_2,F_3)$ *and*

$$\begin{aligned}\mathrm{grad}(f)&\equiv\nabla f={}^t\big(\partial_1 f,\partial_2 f,\partial_3 f\big)\\ \mathrm{div}(F)&\equiv\nabla\cdot F=\partial_1F_1+\partial_2F_2+\partial_3F_3\\ \mathrm{curl}(F)&\equiv\nabla\times F={}^t\big(\partial_2F_3-\partial_3F_2,\partial_3F_1-\partial_1F_3,\partial_1F_2-\partial_2F_1\big)\\ \Delta f&=\partial_1^2f+\partial_2^2f+\partial_3^2f.\end{aligned}$$

Show then the following relations:

(i) $\mathrm{div}(fF)=f\mathrm{div}(F)+\mathrm{grad}(f)\cdot F$,

(ii) $\mathrm{div}\big(\mathrm{curl}(F)\big)=0$,

(iii) $\mathrm{curl}\big(\mathrm{grad}(f)\big)=0$,

(iv) $\mathrm{curl}\big(\mathrm{curl}(F)\big)=\mathrm{grad}\big(\mathrm{div}(F)\big)-\Delta F$, *with* $\Delta F={}^t(\Delta F_1,\Delta F_2,\Delta F_3)$.

In the following exercises we call *a vector field* a function f defined on an open set $\Omega\subset\mathbb{R}^n$ and taking values in $\mathbb{R}^n$. In other words, for $\Omega\subset\mathbb{R}^n$ open, a function $f:\Omega\to\mathbb{R}^d$ is a vector field if $d=n$.

Exercise 2 *Provide a picture for the following vector fields:*

(i) $f:\mathbb{R}^2\to\mathbb{R}^2,\quad f(x,y)=\binom{1}{1}$,

(ii) $g:\mathbb{R}^2\to\mathbb{R}^2,\quad g(x,y)=x\,E_1+y\,E_2$,

(iii) $\nabla k:\mathbb{R}^2\to\mathbb{R}^2$ *for* $k:\mathbb{R}^2\to\mathbb{R}$ *given by* $k(x,y)=\frac{1}{1+x^2+y^2}$.

Let $f:\Omega\to\mathbb{R}^n$ be a vector field. If there exists a differentiable function $\phi:\Omega\to\mathbb{R}$ such that $f=\nabla\phi$ we say that ϕ is a *potential function for* f.

Exercise 3 *For the following functions, does a potential function exist ?*

(i) $f(x,y)=(y,x)$, *(ii)* $f(x,y)=(3x^2y+2x+y^3,x^3+3xy^2-2y)$,

(iii) $f(x,y)=\big(\cos(x),\sin(y)\big)$, *(iv)* $f(x,y,z)=\big(x^2-yz,y^2-zx,z^2-xy\big)$.

Exercise 4 *Consider the vector field* $F:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}^2$ *defined for* $(x,y)\neq(0,0)$ *by*

$$F(x,y)=\Big(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\Big)$$

(i) Represent graphically this vector field (you can use polar coordinates),

(ii) Can you find a potential function for this vector field, and if so exhibit it.