
Homework 4

Exercise 1 (i) Compute the Taylor expansion around $(0, 0)$ and up to the second order of the function

$$\mathbb{R}^2 \ni (x, y) \mapsto e^{x^2+xy+y^2} \in \mathbb{R}.$$

(ii) Compute the Taylor expansion around $(0, 0)$ up to the third order of the function

$$\mathbb{R}^2 \ni (x, y) \mapsto e^{x+y} \in \mathbb{R}.$$

By fixing then $x = y = 1/2$ in the polynomial you have obtained, what can you say about the number e ?

Exercise 2 Consider the functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $\varphi(t) := \begin{pmatrix} \cos(t) \\ t^2 \end{pmatrix}$, and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) := e^{3x+2y}$. We consider the composition of these two functions, namely $F : \mathbb{R} \rightarrow \mathbb{R}$ given by $F = f \circ \varphi$. Compute the derivative of this function by two different methods: once by a direct computation, and once as the derivative of a composed function (chain rule).

So far we have considered the norm in \mathbb{R}^n provided by the formula

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{j=1}^n x_j^2} =: \|X\|_2.$$

We could have considered other expressions, as for example

$$\|X\|_1 := |x_1| + |x_2| + \cdots + |x_n| = \sum_{j=1}^n |x_j|, \quad \text{or} \quad \|X\|_\infty := \max_{j \in \{1, \dots, n\}} |x_j|.$$

Exercise 3 a) Show that the two alternative expressions also define norms. Namely, they satisfy the three conditions of any norm $\|\cdot\|$:

(i) $\|X\| \geq 0$ for any $X \in \mathbb{R}^n$, and $\|X\| = 0$ if and only if $X = \mathbf{0}$,

(ii) $\|\lambda X\| = |\lambda| \|X\|$, for any $\lambda \in \mathbb{R}$ and $X \in \mathbb{R}^n$,

(iii) $\|X + Y\| \leq \|X\| + \|Y\|$, for any $X, Y \in \mathbb{R}^n$.

b) Draw the unit ball of \mathbb{R}^2 with the three norms $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$.