Homework 14

Exercise 1 *Let $f:\mathbb{R}^2\to\mathbb{R}$ be sufficiently many times differentiable and satisfying the Laplace equation*

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$$

(i) Let c be a closed parametric curve oriented counterclockwise and non-intersecting. Show that

$$\int_c \begin{pmatrix}\partial_y f\\ -\partial_x f\end{pmatrix}=0,$$

(ii) Show that $f(0,0)=\frac{1}{2\pi}\int_0^{2\pi} f\big(r\cos(\theta),r\sin(\theta)\big)\mathrm{d}\theta$ *for any* $r>0$.

Exercise 2 (Final exam 2020) *The aim of this exercise is to show that the area of a surface is independent of its parametrization. Consider an open set $\Omega\subset\mathbb{R}^2$, a function $g:\Omega\to\mathbb{R}$ of class C^1, and the parametric surface given by*

$$\Omega\ni(s,t)\mapsto q(s,t)=\begin{pmatrix}s\\ t\\ g(s,t)\end{pmatrix}\in\mathbb{R}^3$$

Let also $\varphi:\Lambda\ni(x,y)\mapsto\big(\varphi_1(x,y),\varphi_2(x,y)\big)\in\Omega$ be a diffeomorphism of class C^1.

(i) Write the formula for the computation of the area of $q(\Omega)$,

(ii) Compute $[\partial_1(g\circ\varphi)](x,y)$ and $[\partial_2(g\circ\varphi)](x,y)$ (altogether you should obtain 4 terms),

(iii) Using the previous information, compute the vectors $[\partial_1(q\circ\varphi)](x,y)$ and $[\partial_2(q\circ\varphi)](x,y)$,

(iv) Compute $\Big([\partial_1(q\circ\varphi)](x,y)\Big)\times\Big([\partial_2(q\circ\varphi)](x,y)\Big)$, and factor the common term $\Big[(\partial_1\varphi_1)(\partial_2\varphi_2)-(\partial_1\varphi_2)(\partial_2\varphi_1)\Big](x,y)$,

(v) Conclude with the general formula for the change of variables for a volume integral that the area of a surface is independent of its parametrization, or more precisely show that the area of $q(\Omega)$ is equal to the area of $[q\circ\varphi](\Lambda)$.

Exercise 3 (Final exam 2018) *Consider the set $\Omega:=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\leq 9\}$ and the surface S defined by*

$$S:=\big\{(x,y,z)\in\mathbb{R}^3\mid z=9-x^2-y^2 \text{ for } (x,y)\in\Omega\big\}.$$

1. *Sketch the surface S,*
2. *Provide a parametrization of the surface S,*
3. *If $\Psi:\mathbb{R}^3\to\mathbb{R}^3$ is the vector field given by $\Psi(x,y,z)=\big(2z-y,x+z,3x-2y\big)$, compute the flux of $\mathrm{curl}(\Psi)$ through the surface S, with the normal vector on S pointing outside of the surface,*
4. *Compute the integral of Ψ along the boundary of S and check Stokes's theorem on this example.*