The following statement has been proposed by Duc Truyen, and the proof has been provided by Chang Sun.

Theorem 0.1 (Strongly connected bipartite graphs)**.** *A strongly connected oriented graph G is bipartite if and only if it has no cycle of odd length.*

Proof of Theorem 0.1. Necessity ($\Rightarrow$): Suppose G is bipartite, then each step in a walk switches between the two bipartitions. Any closed walk (if exists) requires the walk to end on the same side as it started, and this forces the total number of steps to be even.

Sufficiency ($\Leftarrow$): Let G be a graph with at least two vertices and no cycle of odd length. Choose arbitrary $x, y \in V$. Let L_1 be a shortest $y \to x$ path, and let L_2 be a shortest $x \to y$ path. Let a_1 be the first vertex common to both paths from y (go reversely with path L_2), and let a_{k+1} be the first vertex common to both paths from a_k until one reaches $a_n = x$. Let C_k be the cycles obtained by sections of L_1 and L_2 between a_k and a_{k-1}, see Figure 1.

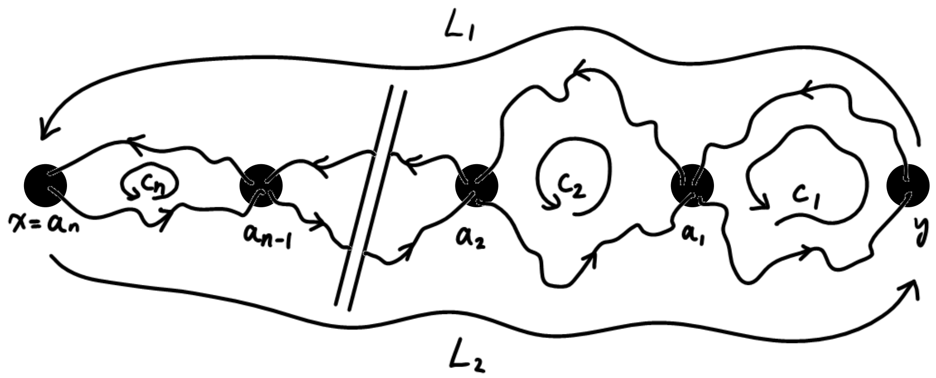

Figure 1: Construction for the sufficiency part

Clearly, the length of the composition of L_1 and L_2 is $\mathrm{d}(y, x) + \mathrm{d}(x, y) = \sum_{k=1}^{n} c_k$ with c_k being the length of C_k. Given that there exists no cycle with odd length, all c_k are even. Thus $\mathrm{d}(y, x) + \mathrm{d}(x, y)$ is even and therefore $\mathrm{d}(y, x) = \mathrm{d}(x, y)$ modulo 2.

Let us now pick a vertex u from G, and define a partition (X_0, X_1) of V as follows

$$X_0 := \{x \in V \mid \mathrm{d}(u, x) = 0 \pmod 2\} = \{x \in V \mid \mathrm{d}(x, u) = 0 \pmod 2\}$$
$$X_1 := \{x \in V \mid \mathrm{d}(u, x) = 1 \pmod 2\} = \{x \in V \mid \mathrm{d}(x, u) = 1 \pmod 2\}.$$

It remains to show that this partition defines a bipartition for G.

If (X_0, X_1) is not a bipartition of G, then there are two vertices in one of the sets, say v and w that are joined by an edge. We denote this edge by e, and assume with no loss of generality that this edge starts at v and ends at w. Let P_1 be a shortest $u \to v$ path, and let P_2 be a shortest $w \to u$ path. By definition of the sets X_0 and X_1, the

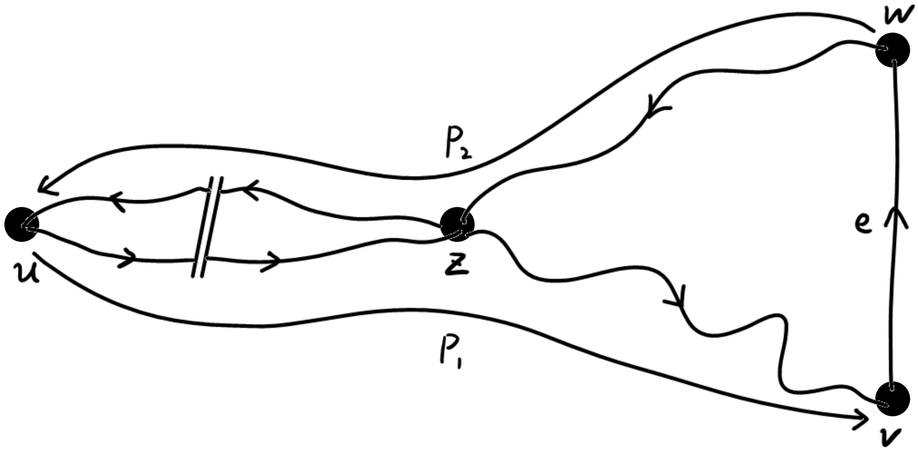

Figure 2: Figure 2 for sufficiency part of Lemma 0.1 proof

length of these paths are both even or both odd. Starting from vertex u, let z be the last vertex common to both paths, see Figure 2.

As both $\mathrm{d}(w,u)$ and $\mathrm{d}(u,v)$ are odd or even, and $\mathrm{d}(u,z) = \mathrm{d}(z,u) \pmod 2$, one has

$$\begin{aligned}
&\mathrm{d}(w,z) + \mathrm{d}(z,v) \\
=&\mathrm{d}(w,u) - \mathrm{d}(z,u) + \mathrm{d}(u,v) - \mathrm{d}(u,z) \pmod 2 \\
=&\big(\mathrm{d}(w,u) + \mathrm{d}(u,v)\big) - \big(\mathrm{d}(u,z) + \mathrm{d}(z,u)\big) \pmod 2 \\
=&0 \pmod 2
\end{aligned}$$

Therefore, the cycle passing from w to z on P_2, then from z to v on P_1, and finally from v to u by e, is of length $\mathrm{d}(w,z) + \mathrm{d}(z,v) + 1 = 1 \pmod 2$. It is thus a cycle of odd length, which contradicts the assumption. Thus, (X_0, X_1) is a bipartition for G. $\square$