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Some problems related to section 7.2 Plane graphs.

Q1: Let $G = (V, E)$ be a connected simple plane graph with $|V(G)| < 12$. Prove that G has a vertex of degree at most 4.

Proof:

If $|V(G)| \leq 5$ then the maximal degree $\Delta(G)$ is at most 4. Thus, every vertex in G has degree at most 4.

In the case $5 < |V(G)| < 12$, since each edge adds two degree to the total degree of vertices and also based on theorem 7.17, one has:

$$\sum_{v \in V(G)} \deg(v) = 2\,|E(G)| \leq 2(3|V(G)| - 6) < 6|V(G)| - |V(G)| = 5|V(G)| \quad (*)$$

If every vertex has degree more than 4 (or equivalently, at least 5), then

$\sum_{v \in V(G)} \deg(v) \geq 5|V(G)|$, which contradicts (*).

Hence, the graph G has a vertex of degree at most 4.

Q2: Prove that for any three vertices x, y, z of a simple plane graph $G = (V, E)$ with number of vertices at least 3, the sum of the degrees $deg(x) + deg(y) + deg(z)$ is at most $2|V(G)| + 2$.

Proof: The sum $deg(x) + deg(y) + deg(z)$ comprises in the connection among the vertices x, y, z themselves, and the connection between x, y, z and other vertices in G. If x, y, z are pairwise connected, the three pairwise edges among them make the sum $deg(x) + deg(y) + deg(z)$ increase by 6. Since G cannot have $K_{3,3}$ as a subgraph, at most two vertices in $V \backslash \{x, y, z\}$ can be connected to all the three vertices x, y, z. With the exception of possibly two vertices, all other $(|V(G)| - 5)$ vetices in $V \backslash \{x, y, z\}$ are adjacent to at most 2 vertices among x, y, z. Then,

$$deg(x) + deg(y) + deg(z) \leq 6 + 3 \times 2 + 2(|V(G)| - 5) = 2|V(G)| + 2 \quad \text{(Q.E.D)}$$

Definition (**complete bipartite graph**): The graph $G = (V, E)$ is a **complete bipartite graph** if it is a bipartite graph whose vertices can be partitioned into two subsets V_1 and V_2 such that: (1) no edge in E has both endpoints in the same subset; (2) any pair of vertices $v_1 \in V_1$ and $v_2 \in V_2$ are adjacent, or equivalently, connected by an edge in E.

With the partitions of size $|V_1| = r$ and $|V_2| = s$, the complete bipartite graph is denoted by $K_{r,s}$.

Next, we are going to mentioned some examples of complete bipartite graph.

(1) Star graphs:

For any positive integer s, the graph $K_{1,s}$ is called a *star*. All complete bipartite graphs which are trees are stars. In addition, the graph $K_{1,3}$ is also called a *claw*.

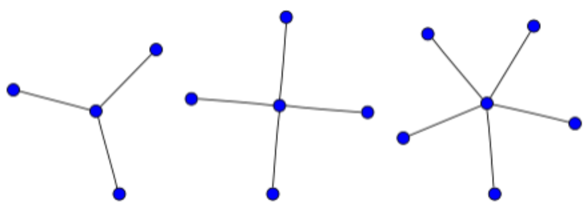

Figure 1: The star graphs $K_{1,3}$,$K_{1,4}$ and $K_{1,5}$

(2) $K_{n,n}$ graphs (n is a positive integer):

The graph $K_{3,3}$ is also called *utility graph*. As mentioned already, any plane graph cannot contain $K_{3,3}$ as a subgraph.

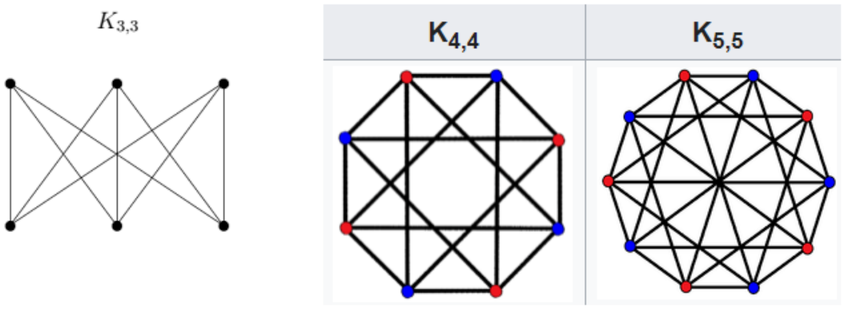

Figure 2: The complete bipartite graphs $K_{3,3}$, $K_{4,4}$ and $K_{5,5}$

However, not every $K_{r,s}$ graph is plane graph.

Q3: Determine all positive integers r and s for which $K_{r,s}$ is a plane graph.

The answer is $(r, s) = (1, s)\ \forall s \in \mathrm{Z}^+$ and $(r, s) = (2, s)\ \forall s \in \mathrm{Z}^+$.

Explanation: If $r > 3$ and $s > 3$ then $K_{r,s}$ has $K_{3,3}$ as its subgraph and thus it is not a plane graph. Meanwhile, $K_{1,s}$ and $K_{2,s}$ can be plane graphs for any positive integer s. The graphs $K_{1,s}$ (also known as star graphs) can be drawn with one vertex in the center, surrounded by s vertices; the graphs $K_{2,s}$ can be drawn with s vertices on a line in the plane and the other two vertices, one on each side of this line.

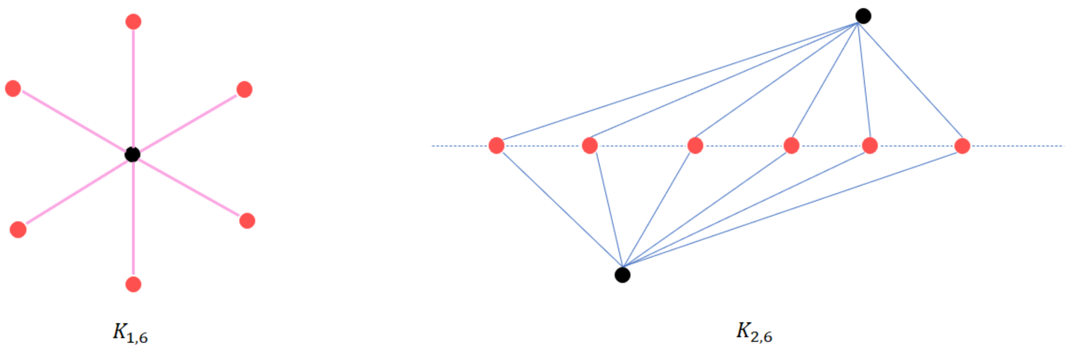

Figure 3: An example of $K_{1,s}$ and $K_{2,s}$ with $s = 6$

References

[1] https://en.wikipedia.org/wiki/Complete_bipartite_graph#:~:text=5%20References-,Definition,is%20part%20of%20the%20graph.