Examples of the arc length parametrization

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This is related to

- Chapter 2 in Calculus II
- Chapter 2 in O'Neill

0.0.1. Example 1 (A straight line) Let v be a vector such that $\|v\| = r > 0$ in $\mathbb{R}^2$ and p be a point in $\mathbb{R}^2$. Let $f : [a, b] \to \mathbb{R}^2$ be a parametric curve such that $f(t) = tv + p$ for t in $[a, b] \subset \mathbb{R}$. f is of class C^1 on (a, b) and regular on (a, b). The length L_f of the corresponding curve $f((a, b))$ is

$$L_f := \int_a^b \|f'(t)\|dt = \int_a^b \|v\|dt = (b - a)r.$$

Let $\psi : (a, b) \to \mathbb{R}$ be a function given for any $t \in (a, b)$ by

$$\psi(t) := \int_a^t \|f'(s)\|ds = (t - a)r.$$

ψ is strictly increasing on $[a, b]$ and differentiable, with $\psi'(t) = \|f'(t)\| = \|v\| = r > 0$ for any $t \in (a, b)$. ψ has image equal to $[0, L_f]$. It follows that ψ has an inverse $\psi^{-1} : [0, L_f] \to [a, b]$.

In this example, $s = \psi(t) = rt - ar$. Thus $t = \psi^{-1}(s) = \dfrac{s + ar}{r} = \dfrac{s}{r} + a$.

ψ^{-1} is differentiable on $(0, L_f)$ and its derivative is given for any $s \in (0, L_f)$ by

$$\psi^{-1}(s)' = \frac{1}{\psi'(\psi^{-1}(s))} = \frac{1}{\|f'(\psi^{-1}(s))\|} = \frac{1}{\|v\|} = \frac{1}{r}.$$

Thus, if we set

$$\varphi : [0, L_f] \to [a, b], \quad \varphi(s) := \psi^{-1}(s),$$

then φ is a diffeomorphism of class C^1 on $(0, L_f)$, and the composed map $f \circ \varphi : [0, L_f] \to \mathbb{R}^2$ is

$$(f \circ \varphi)(s) = f(\varphi(s)) = (\frac{s}{r} + a)v + p = \frac{v}{r}s + av + p.$$

This satisfies

$$(f \circ \varphi)'(s) = f'(\varphi(s))\varphi'(s) = f'(\varphi(s))\frac{1}{\|f'(\psi^{-1}(s))\|} = v\frac{1}{\|v\|} = \frac{1}{r}v.$$

with norm

$$\|(f \circ \varphi)'(s)\| = \left\| f'(\varphi(s)) \frac{1}{\|f'(\varphi(s))\|} \right\| = \frac{1}{\|f'(\varphi(s))\|} \|f'(\varphi(s))\| = 1.$$

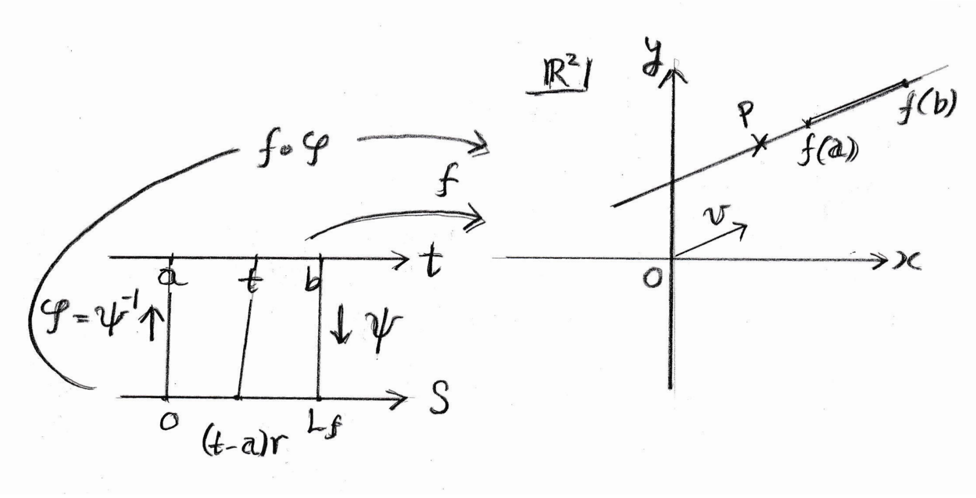

FIG. 0.1 Arc length parametrization of a straightline

0.0.2. Example 2 (A circle) Let $f : [0, 2\pi] \to \mathbb{R}^2$ be a parametric curve such that $f(t) = (r\cos(t), r\sin(t))$ for t in $[0, 2\pi] \subset \mathbb{R}$ and for some number $r > 0$. f is of class C^1 on $(0, 2\pi)$ and regular on $(0, 2\pi)$. The length L_f of the corresponding curve $f((0, 2\pi))$ is

$$L_f := \int_0^{2\pi} \|f'(t)\| dt = \int_0^{2\pi} r dt = 2\pi r.$$

Let $\psi : (0, 2\pi) \to \mathbb{R}$ be a function given for any $t \in (0, 2\pi)$ by

$$\psi(t) := \int_0^t \|f'(s)\| ds = rt.$$

ψ is strictly increasing on $[0, 2\pi]$ and differentiable, with $\psi'(t) = \|f'(t)\| = r > 0$ for any $t \in (0, 2\pi)$. ψ has image equal to $[0, L_f]$. It follows that ψ has an inverse $\psi^{-1} : [0, L_f] \to [0, 2\pi]$.

In this example, $s = \psi(t) = rt$. Thus $t = \psi^{-1}(s) = \frac{s}{r}$.

ψ^{-1} is differentiable on $(0, L_f)$ and its derivative is given for any $s \in (0, L_f)$ by

$$\psi^{-1}(s)' = \frac{1}{\psi'(\psi^{-1}(s))} = \frac{1}{\|f'(\psi^{-1}(s))\|} = \frac{1}{\|(-r\sin(\frac{s}{r}), r\cos(\frac{s}{r})\|} = \frac{1}{r}.$$

Thus, if we set

$$\varphi : [0, L_f] \to [0, 2\pi], \quad \varphi(s) := \psi^{-1}(s),$$

then φ is a diffeomorphism of class C^1 on $(0, L_f)$, and the composed map $f \circ \varphi : [0, L_f] \to \mathbb{R}^2$ is

$$(f \circ \varphi)(s) = f(\varphi(s)) = (r\cos(\frac{s}{r}), r\sin(\frac{s}{r})).$$

This satisfies

$$(f \circ \varphi)'(s) = f'(\varphi(s))\varphi'(s) = f'(\varphi(s))\frac{1}{\|f'(\psi^{-1}(s))\|} = (-r\sin(\frac{s}{r}), r\cos(\frac{s}{r}))\frac{1}{r} = (-\sin(\frac{s}{r}), \cos(\frac{s}{r})).$$

with norm

$$\|(f \circ \varphi)'(s)\| = \left\|f'(\varphi(s))\frac{1}{\|f'(\varphi(s))\|}\right\| = \frac{1}{\|f'(\varphi(s))\|}\|f'(\varphi(s))\| = \frac{1}{r}r = 1.$$