Extension 6.3.4 Naohiro Tsuzu

Consider an infinite dim separable Hilbert sp. $\mathcal{H}$.

Recall

- A Fredholm op. T on $\mathcal{H}$ is a bdd. op. on $\mathcal{H}$ which satisfies that

  $T\mathcal{H}$ is closed., $\dim \operatorname{Ker} T < \infty$, and $\dim \operatorname{Ker} T^* < \infty$

- The Fredholm index of a Fredholm op. T is defined by

  $\operatorname{index}(T) = \dim \operatorname{Ker} T - \dim \operatorname{Ker} T^* \in \mathbb{Z}$.

- $\exists\ \alpha : K_0(K(\mathcal{H})) \longrightarrow \mathbb{Z}$ isom. s.t.

  $\alpha([P]_0) = \operatorname{Tr}(P)$ for each proj. P in $K(\mathcal{H})$

Prop. For each Fredholm op. T on $\mathcal{H}$,

$\operatorname{index}(T) = (\alpha \circ \delta_1)([\pi(T)]_1)$

where δ_1 is the index map associated with

$$0 \longrightarrow K(\mathcal{H}) \xhookrightarrow{\iota} B(\mathcal{H}) \xrightarrow{\pi} Q(\mathcal{H}) \longrightarrow 0$$

Proof.

Let $T = S|T|$ be the polar decomposition of T.

Set $A := \pi(T)$, $U := \pi(S)$

By Atkinson's thm, A is inv. in $Q(\mathcal{H})$.

Since S is partial isometry,

$E := \mathbb{1} - S^*S$, $F := \mathbb{1} - SS^*$ are projs

For $t \in [0,1]$, set $R_t := S(t|T| + (1-t)\mathbb{1})$

For each $t \in [0,1]$,

$\pi(R_t) = U(t|A| + (1-t)\mathbb{1})$ is inv. in $Q(\mathcal{H})$

Therefore R_t is Fredholm.

Then $A \sim_h U$ in $GL(Q(\mathcal{H}))$

because $t \mapsto \pi(R_t)$ is conti. path in $GL(Q(\mathcal{H}))$

We obtain that $\text{index}(T) = \text{index}(S)$, Moreover, we get $[A]_1 = [U]_1$

by extending $[\cdot]_1 : \mathcal{U}_\infty(Q(\mathcal{H})) \to K_1(Q(\mathcal{H}))$

to a map $[\cdot]_1 : GL_\infty(Q(\mathcal{H})) \to K_1(Q(\mathcal{H}))$.

Then (6.6) yields that

$$
\begin{aligned}
(\alpha \circ \delta_1)([\pi(T)]_1) &= (\alpha \circ \delta_1)([A]_1) \\
&= (\alpha \circ \delta_1)([U]_1) \\
&= \alpha([E]_0 - [F]_0) \\
&= \mathrm{Tr}(E) - \mathrm{Tr}(F) \\
&= \mathrm{rank}(E) - \mathrm{rank}(F) \\
&= \dim \mathrm{Ker}\, S - \dim \mathrm{Ker}\, S^* \quad \cdots (*) \\
&= \mathrm{index}(S) = \mathrm{index}(T).
\end{aligned}
$$

I used $E\mathcal{H} = \mathrm{Ker}\, S$, $F\mathcal{H} = \mathrm{Ker}\, S^*$ in $(*)$

$\square$