Exercise 3.3.3 Naohiro Tsuzu

Prop. 3.3.2 Let $\mathcal{C}$ and $\mathcal{Q}$ be unital C^*-alg^s.

(i) If $\varphi, \psi : \mathcal{C} \to \mathcal{Q}$ are homotopic $*$-homoms,
then $K_0(\varphi) = K_0(\psi)$

(ii) If $\mathcal{C}$ and $\mathcal{Q}$ are homotopy equiv., then $K_0(\mathcal{C})$ is isom. to $K_0(\mathcal{Q})$. More specifically, $\mathcal{C} \xrightarrow{\varphi} \mathcal{Q} \xrightarrow{\psi} \mathcal{C}$ is a homotopy between $\mathcal{C}$ and $\mathcal{Q}$, then
$K_0(\varphi) : K_0(\mathcal{C}) \to K_0(\mathcal{Q})$ and $K_0(\psi) : K_0(\mathcal{Q}) \to K_0(\mathcal{C})$
are isom. with $K_0(\varphi)^{-1} = K_0(\psi)$

proof

Let F be a path of $*$-homom^s $t \mapsto F(t)$ with $F(0) = \varphi$, $F(1) = \psi$ s.t. $\forall a \in \mathcal{C}$,
$[0,1] \ni t \mapsto F(t)a \in \mathcal{Q}$ is conti.

For each $n \in \mathbb{N}$, we extend F to a ptwise conti. path of $*$-homom^S $F(t): M_n(\mathcal{C}) \to M_n(\mathcal{Q})$

Then $\forall p \in P_n(\mathcal{C})$, $t \mapsto F(t)p$ is conti.

We obtain

$$\varphi(p) = F(0)p \sim_h F(1)p = \psi(p).$$

$$K_0(\varphi)[p]_0 = [\varphi(p)]_0 = [\psi(p)]_0$$
$$= K_0(\psi)[p]_0$$

Therefore, $K_0(\varphi) = K_0(\psi)$

(ii) Combining (i), Prop.3.3.1 (i) and (ii),

$$K_0(\psi) \circ K_0(\varphi)[p]_0$$
$$= K_0(\psi \circ \varphi)[p]_0$$
$$= K_0(id_{\mathcal{C}})[p]_0$$
$$= id_{K_0(\mathcal{C})}[p]_0 = [p]_0.$$

Similarly $K_0(\varphi) \circ K_0(\psi) = id_{K_0(\mathcal{Q})}$ $\square$