Calculus Ⅱ; Proof of Theorem 3.20.
- Differentiability and partial derivatives -

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Theorem 3.20.(Lecture note./ [1,Theorem 3.6, p.304]) Let $\Omega \subset \mathbb{R}^n$ be open, and let $f : \Omega \to \mathbb{R}$. Assume that the partial derivatives of f exist and are continuous on Ω. Then f is differentiable on Ω.

(Proof) We show the proof for the case $n = 2$ at a point $X_0 = (x_{10}, x_{20})$ in $\Omega \subset \mathbb{R}^2$. Let $X = (x_1, x_2)$ be any point in Ω near X_0. Since there exist the partial derivatives of f at X_0, from Definition 3.12 (p.21),

$$\partial_1 f(X_0) = \lim_{h_1 \to 0} \frac{f(x_{10} + h_1, x_{20})}{h_1},$$
$$\partial_2 f(X_0) = \lim_{h_2 \to 0} \frac{f(x_{10}, x_{20} + h_2)}{h_2}.$$

Since $\partial_1 f$ and $\partial_2 f$ are continuous on Ω, from the Mean value theorem, there exist ξ_1 and ξ_2 such that $x_1 < \xi_1 < x_{10}$ or $x_{10} < \xi_1 < x_1$, and $x_2 < \xi_2 < x_{20}$ or $x_{20} < \xi_2 < x_2$, which satisfy

$$f(X) - f(X_0) = f(x_1, x_2) - f(x_{10}, x_{20}) \tag{1}$$
$$= (f(x_1, x_2) - f(x_{10}, x_2)) + (f(x_{10}, x_2) - f(x_{10}, x_{20})) \tag{2}$$
$$= \partial_1 f(\xi_1, x_2)(x_1 - x_{10}) + \partial_2 f(x_{10}, \xi_2)(x_2 - x_{20}). \tag{3}$$

We subtract $\partial_1 f(x_{10}, x_{20})(x_1 - x_{10}) + \partial_2 f(x_{10}, x_{20})(x_2 - x_{20})$ from the both sides of the above equation.

$$f(X) - f(X_0) - \partial_1 f(x_{10}, x_{20})(x_1 - x_{10}) - \partial_2 f(x_{10}, x_{20})(x_2 - x_{20}) \tag{4}$$
$$= \partial_1 f(\xi_1, x_2)(x_1 - x_{10}) + \partial_2 f(x_{10}, \xi_2)(x_2 - x_{20}) - \partial_1 f(x_{10}, x_{20})(x_1 - x_{10}) - \partial_2 f(x_{10}, x_{20})(x_2 - x_{20}) \tag{5}$$
$$= (\partial_1 f(\xi_1, x_2) - \partial_1 f(x_{10}, x_{20}))(x_1 - x_{10}) + (\partial_2 f(x_{10}, \xi_2) - \partial_2 f(x_{10}, x_{20})(x_2 - x_{20}) \tag{6}$$

We denote $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $X_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$, then $(X - X_0) = \begin{pmatrix} x_1 - x_{10} \\ x_2 - x_{20} \end{pmatrix}$

$$= \begin{pmatrix} \partial_1 f(\xi_1, x_2) - \partial_1 f(x_{10}, x_{20}) \\ \partial_2 f(x_{10}, \xi_2) - \partial_2 f(x_{10}, x_{20} \end{pmatrix} \bullet (X - X_0) \tag{7}$$

We divide both sides of the above equation by $\|X - X_0\|$ and make X approach X_0. Then

$$\lim_{X\to X_0} \frac{f(X) - f(X_0) - \begin{pmatrix}\partial_1 f(x_{10}, x_{20})\\ \partial_2 f(x_{10}, x_{20})\end{pmatrix} \bullet (X - X_0)}{\|X - X_0\|} \tag{8}$$

$$= \lim_{X\to X_0} \frac{\begin{pmatrix}\partial_1 f(\xi_1, x_2) - \partial_1 f(x_{10}, x_{20})\\ \partial_2 f(x_{10}, \xi_2) - \partial_2 f(x_{10}, x_{20}\end{pmatrix} \bullet (X - X_0)}{\|X - X_0\|} \tag{9}$$

$$= \lim_{X\to X_0} \begin{pmatrix}\partial_1 f(\xi_1, x_2) - \partial_1 f(x_{10}, x_{20})\\ \partial_2 f(x_{10}, \xi_2) - \partial_2 f(x_{10}, x_{20}\end{pmatrix} \bullet \frac{(X - X_0)}{\|X - X_0\|} \tag{10}$$

From the continuity of $\partial_1 f$ and $\partial_2 f$ at $X_0 \in \Omega$,

$$\lim_{X\to X_0} \partial_1 f(\xi_1, x_2) = \partial_1 f(x_{10}, x_{20}), \quad \lim_{X\to X_0} \partial_2 f(x_{10}, \xi_2) = \partial_1 f(x_{10}, x_{20}).$$

Thus $\lim_{X\to X_0} \begin{pmatrix}\partial_1 f(\xi_1, x_2) - \partial_1 f(x_{10}, x_{20})\\ \partial_2 f(x_{10}, \xi_2) - \partial_2 f(x_{10}, x_{20}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. And $\frac{(X - X_0)}{\|X - X_0\|}$ is a vector of norm 1. Hence

$$\lim_{X\to X_0} \begin{pmatrix}\partial_1 f(\xi_1, x_2) - \partial_1 f(x_{10}, x_{20})\\ \partial_2 f(x_{10}, \xi_2) - \partial_2 f(x_{10}, x_{20}\end{pmatrix} \bullet \frac{(X - X_0)}{\|X - X_0\|} = 0.$$

Therefore

$$\lim_{X\to X_0} \frac{f(X) - f(X_0) - \begin{pmatrix}\partial_1 f(x_{10}, x_{20})\\ \partial_2 f(x_{10}, x_{20})\end{pmatrix} \bullet (X - X_0)}{\|X - X_0\|} \tag{11}$$

$$= \lim_{X\to X_0} \frac{f(X) - f(X_0) - \nabla f(X_0) \bullet (X - X_0)}{\|X - X_0\|} \tag{12}$$

$$= 0 \tag{13}$$

Therefore, from Definition 3.18 (p.24), the above function f on $\Omega \subset \mathbb{R}^2$ is differentiable at a point $X_0 \in \Omega$. We can extend $n = 2$ to arbitrary n straightforward, and we can choose any $X_0 \in \Omega \subset \mathbb{R}^n$ by the way of this proof. As the result the function f on $\Omega \subset \mathbb{R}^n$ is differentiable on Ω, if its partial derivatives exist and are continuous on Ω.

□

References

[1] E.Hairer and G.Wanner, Analysis by its History, Springer 2008.