About Unitization of C^*-algebras

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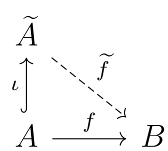
Definition 1. Let A be a C^*-algebra with or without a unit. Its (smallest) unitization $\widetilde{A}$ is any unital C^*-algebra containing A as an ideal and satisfying the following universal property: for any unital C^*-algebra B and any $*$-homomorphism $f\colon A \to B$, there exists a unique unital $*$-homomorphism $\widetilde{f}\colon \widetilde{A} \to B$ with $\widetilde{f} \circ \iota = f$, where $\iota\colon A \to \widetilde{A}$ is the inclusion map.



Proposition 2. For any C^*-algebra A with or without a unit, there exists its unitization $\widetilde{A}$, and $\widetilde{A}$ is unique up to an isomorphism.

Proof. First we show the uniqueness. Let $\widetilde{A}$ and $\widetilde{A}'$ be two unitizations of A. By the universal property, there exist $\phi\colon \widetilde{A} \to \widetilde{A}'$ and $\psi\colon \widetilde{A}' \to \widetilde{A}$ with $\phi \circ \iota_{\widetilde{A}} = \iota_{\widetilde{A}'}$ and $\psi \circ \iota_{\widetilde{A}'} = \iota_{\widetilde{A}}$. Then $\psi \circ \phi\colon \widetilde{A} \to \widetilde{A}$ and $\phi \circ \psi\colon \widetilde{A}' \to \widetilde{A}'$ satisfy $\psi \circ \phi \circ \iota_{\widetilde{A}} = \iota_{\widetilde{A}}$ and $\phi \circ \psi \circ \iota_{\widetilde{A}'} = \iota_{\widetilde{A}'}$. On the other hand, $\mathrm{id}_{\widetilde{A}}\colon \widetilde{A} \to \widetilde{A}$ and $\mathrm{id}_{\widetilde{A}'}\colon \widetilde{A}' \to \widetilde{A}'$ satisfy $\mathrm{id}_{\widetilde{A}} \circ \iota_{\widetilde{A}} = \iota_{\widetilde{A}}$ and $\mathrm{id}_{\widetilde{A}'} \circ \iota_{\widetilde{A}} = \iota_{\widetilde{A}}$. By the uniqueness in the universal property, $\phi \circ \psi = \mathrm{id}_{\widetilde{A}}$ and $\psi \circ \phi = \mathrm{id}_{\widetilde{A}'}$. Then $\phi\colon \widetilde{A} \to \widetilde{A}'$ is a $*$-isomorphism with $\phi \circ \iota_{\widetilde{A}} = \iota_{\widetilde{A}'}$. Then the unitization is unique up to an isomorphism. Next we show the existence. The case A is unital is clear. Then we consider the case A is non-unital. Set $\widetilde{A} = A \oplus \mathbb{C}$ as a direct sum of complex vector spaces and define the multiplication and involution on $\widetilde{A}$ by

$$(a, \alpha) \cdot (b, \beta) = (ab + \beta a + \alpha b, \alpha\beta), \quad (a, \alpha)^* = (a^*, \overline{\alpha}).$$

Let $\| \cdot \|_A$ be the norm of A. For each $x = (a, \alpha) \in \widetilde{A}$,

$$|||x|||_{\widetilde{A}} = \sup_{b \in A, \|b\|_A \le 1} \|ab + \alpha b\|_A,$$
$$\|x\|_{\widetilde{A}} = \max\{|||x|||_{\widetilde{A}}, |\alpha|\}.$$

Then $(\widetilde{A}, \|\cdot\|_{\widetilde{A}})$ is a unital C^*-algebra with unit $1_{\widetilde{A}} = (0, 1)$. We define the *-homomorphisms $\iota\colon A \to \widetilde{A}$, $\pi\colon \widetilde{A} \to \mathbb{C}$, and $\lambda\colon \mathbb{C} \to \widetilde{A}$ by

$$\iota(a) = (a, 0), \quad \pi(a, \alpha) = \alpha, \quad \lambda(\alpha) = (0, \alpha).$$

Then the following sequence is a split short exact sequence of C^*algebras:

$$0 \to A \xrightarrow{\iota} \widetilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0.$$

This construction of $\widetilde{A}$ is valid even when A is unital, and $\widetilde{A}$ is isomorphic to $A \oplus \mathbb{C}$ as C^*-algebras if and only if A is unital. But $\widetilde{A}$ is not the smallest unitization. We will prove that this construction is well-defined in the last part of this report. By identifying ι with the inclusion map and identifying π with the quotient map, $\widetilde{A}$ is the unitization of A. Indeed, for any unital C^*-algebra B with unit 1_B and any *-homomorphism $f\colon A \to B$, we define $\widetilde{f}\colon \widetilde{A} \to B$ by $\widetilde{f}(a, \alpha) = a + \alpha 1_B$. Then $\widetilde{f}$ is clearly a unital *-homomorphism with $\widetilde{f} \circ \iota = f$. If $\widetilde{f}'\colon \widetilde{A} \to B$ is another unital *-homomorphism with $\widetilde{f}' \circ \iota = f$, then for any $(a, \alpha) \in \widetilde{A}$,

$$\widetilde{f}'(a, \alpha) = \widetilde{f}' \circ \iota(a) + \alpha 1_B = f(a) + \alpha 1_B = \widetilde{f} \circ \iota(a) + \alpha 1_B = \widetilde{f}(a, \alpha).$$

Then $\widetilde{f}' = \widetilde{f}$ and so $\widetilde{f}$ is unique. □

By the universal property, the following holds.

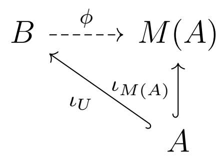
Corollary 3. For any *-homomorphism $f\colon A \to B$, there exists a unique unital *-homomorphism $\widetilde{f}\colon \widetilde{A} \to \widetilde{B}$ such that $\widetilde{f} \circ \iota_A = \iota_B \circ f$. In other words, the unitization of C^*-algebras is functorial.

Example 4. Let X be a non-compact and locally compact Hausdorff space. Then $(C_0(X), \|\cdot\|_\infty)$ is a non-unital C^*-algebra. The unitization of $(C_0(X), \|\cdot\|_\infty)$ is isomorphic to $(C(X^+), \|\cdot\|_\infty)$ where X^+ is the one-point compactification of X. In this sense, the unitization of C^*-algebras is the non-commutative analogue of one-point compactification of non-compact and locally compact Hausdorff spaces.

We considered the smallest unitization of C^*-algebras so far. But there exists a notion of "the largest" unitization of C^*-algebras. We introduce it without proof.

Definition 5. Let A be a C^*-algebra. An ideal I of A is essential if $I \cap J \neq 0$ for any non-zero ideal J of A.

Definition 6. Let A be C^*-algebra. Its multiplier algebra $M(A)$ is any unital C^*-algebra containing A as an essential ideal and satisfying the following universal property: for any C^*-algebra B containing A as an essential ideal, there exists a unique *-homomorphism $\phi\colon B \to M(A)$ such that $\phi \circ \iota_B = \iota_{M(A)}$ and where $\iota_B\colon A \to B$ and $\iota_{M(A)}\colon A \to M(A)$ are the inclusion maps.



Proposition 7. For any C^*-algebra A with or without a unit, there exists its multiplier $M(A)$, and $M(A)$ is unique up to an isomorphism.

The multiplier algebra $M(A)$ can be constructed as the set of double centralizer of A (see [5]).

Example 8. Let X be a locally compact Hausdorff space. Let $A = (C_0(X), \|\cdot\|_\infty)$ be a C^*-algebra. Then $M(A)$ is isomorphic to $(C_b(X), \|\cdot\|_\infty)$. On the other hand, $C_b(X)$ is isomorphic to $C(\beta X)$ where βX is the Stone-Čech compactification of X. In this sense, the multiplier of C^*-algebras is the non-commutative analogue of the Stone-Čech compactification of locally compact Hausdorff spaces.

Example 9. Let $\mathcal{H}$ be a separable Hilbert space and let $A = \mathcal{K}(\mathcal{H})$ be the C^*-algebra of all compact operators on $\mathcal{H}$. Then $M(A)$ is isomorphic to $\mathcal{B}(\mathcal{H})$, the C^*-algebra of all bounded operators on $\mathcal{H}$.

Finally, we prove that the unitization $\widetilde{A}$ of A constructed above is a C^*-algebra.

Proof. First, we show that $\|\cdot\|_{\widetilde{A}}$ is a norm. We only show that if $\|x\|_{\widetilde{A}} = 0$, then $x = 0$. The rest of the condition for the norm is clear since $\|\cdot\|_A, |\cdot|$ are norms. If $x = (a, \alpha) \in \widetilde{A}$ with $\|x\|_{\widetilde{A}} = 0$, then $|||x|||_{\widetilde{A}} = 0$ and $|\alpha| = 0$. So, $\alpha = 0$ and $\|ab\|_A = 0$ for any $b \in A$ with $\|b\|_A \leq 1$. By the C^*-condition of $\|\cdot\|_A$, if $a \neq 0$, $\|a^*/\|a\|_A\|_A = 1$ and

$$\|a\|_A = \frac{\|a\|_A^2}{\|a\|_A} = \frac{\|aa^*\|_A}{\|a\|_A} = \left\| a\left(\frac{a^*}{\|a\|_A}\right)\right\|_A = 0.$$

Since $\|\cdot\|_A$ is a norm, we have $a = 0$. Thus $x = 0$. Therefore $\|\cdot\|_{\widetilde{A}}$ is a norm.

Next, we show that $\widetilde{A}$ is complete with respect to $\|\cdot\|_{\widetilde{A}}$. Let $\{x_n = (a_n, \alpha_n)\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widetilde{A}$. Then,

$$\begin{aligned}
|\alpha_n - \alpha_m| &\leq \|x_n - x_m\|_{\widetilde{A}} \to 0 \quad (n, m \to \infty),\\
\|a_n - a_m\|_A &= \|(a_n - a_m)(a_n - a_m)^*\|_A^{1/2}\\
&= \begin{cases} \|a_n - a_m\|_A^{1/2} \left\|(a_n - a_m)\left(\frac{(a_n-a_m)^*}{\|a_n-a_m\|_A}\right)\right\|_A^{1/2} & \text{if } a_n - a_m \neq 0\\ 0 & \text{if } a_n - a_m = 0\end{cases}\\
&\leq \|a_n - a_m\|_A^{1/2} \sup_{\|b\|_A\leq 1} \|(a_n - a_m)b\|_A^{1/2}\\
&= \|a_n - a_m\|_A^{1/2} |||(a_n - a_m, 0)|||_{\widetilde{A}}^{1/2},
\end{aligned}$$

so, one has

$$\begin{aligned}
\|a_n - a_m\|_A &\le |||(a_n - a_m, 0)|||_{\widetilde{A}} \\
&\le \|(a_n - a_m, 0)\|_{\widetilde{A}} \\
&\le \|x_n - x_m\|_{\widetilde{A}} + \|(0, \alpha_n - \alpha_m)\|_{\widetilde{A}} \\
&= \|x_n - x_m\|_{\widetilde{A}} + |\alpha_n - \alpha_m| \to 0 \quad (n, m \to \infty).
\end{aligned}$$

Then $\{a_n\}_{n\in A}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in A and $\mathbb{C}$ respectively. Since A and $\mathbb{C}$ are complete, there exist $a \in A$ and $\alpha \in \mathbb{C}$ such that $\|a_n - a\|_A \to 0$ and $|\alpha_n - \alpha| \to 0$ as $n \to \infty$. Since

$$\begin{aligned}
|||x_n - x|||_{\widetilde{A}} &= \sup_{\|b\|_A \le 1} \|(a_n - a)b + (\alpha_n - \alpha)b\|_A \\
&\le \sup_{\|b\|_A \le 1} (\|a_n - a\|_A\|b\|_A + |\alpha_n - \alpha|\|b\|_A) \\
&= \|a_n - a\|_A + |\alpha_n - \alpha|,
\end{aligned}$$

then, one has

$$\begin{aligned}
\|x_n - x\|_{\widetilde{A}} &= \|(a_n - a, \alpha_n - \alpha)\|_{\widetilde{A}} \\
&= \max\{|||x_n - x|||_{\widetilde{A}}, |\alpha_n - \alpha|\} \\
&\le \max\{\|a_n - a\|_A + |\alpha_n - \alpha|, |\alpha_n - \alpha|\} \to 0 \quad (n \to \infty).
\end{aligned}$$

Thus $\widetilde{A}$ is complete with respect to $\|\cdot\|_{\widetilde{A}}$.

Next we show that $(\widetilde{A}, \|\cdot\|_{\widetilde{A}})$ is a Banach algebra. For any $x = (a, \alpha), y = (b, \beta), z = (c, \gamma) \in \widetilde{A}$ and $\zeta \in \mathbb{C}$,

$$\begin{aligned}
(\zeta x)y &= (\zeta a, \zeta\alpha)(b, \beta) = (\zeta ab + \beta\zeta a + \alpha\zeta b, \alpha\beta), \\
\zeta(xy) &= \zeta(ab + \beta a + \alpha b, \alpha\beta) = (\zeta ab + \zeta\beta a + \zeta\alpha b, \zeta\alpha\beta), \\
x(\zeta y) &= (a, \alpha)(\zeta b, \zeta\beta) = (\zeta ab + \beta\zeta a + \alpha\zeta b, \alpha\beta).
\end{aligned}$$

Then $(\zeta a)b = \zeta(ab) = a(\zeta b)$.

$$\begin{aligned}
x(y + z) &= (a, \alpha)(b + c, \beta + \gamma) \\
&= (a(b + c) + (\beta + \gamma)a + \alpha(b + c), \alpha(\beta + \gamma)) \\
&= ((ab + \beta a + \alpha b) + (ac + \gamma a + \alpha c), \alpha\beta + \alpha\gamma) \\
&= xy + xz. \\
(x + y)z &= (a + b, \alpha + \beta)(c, \gamma) \\
&= ((a + b)c + \gamma(a + b) + (\alpha + \beta)c, (\alpha + \beta)\gamma)) \\
&= ((ac + \gamma a + \alpha c) + (bc + \gamma b + \beta c), \alpha\gamma + \beta\gamma) \\
&= xz + yz.
\end{aligned}$$

Then $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$. Since

$$\begin{aligned}
|||xy|||_{\widetilde{A}} &= |||(a,\alpha)(b,\beta)|||_{\widetilde{A}} \\
&= |||(ab + \beta a + \alpha b, \alpha\beta)|||_{\widetilde{A}} \\
&= \sup_{\|c\|_A \leq 1} \|(ab + \beta a + \alpha b)c + \alpha\beta c\|_A \\
&= \sup_{\|c\|_A \leq 1} \|a(bc + \beta c) + \alpha(bc + \beta c)\|_A \\
&\leq |||x|||_{\widetilde{A}} \sup_{\|c\|_A \leq 1} \|bc + \beta c\|_A \\
&\leq |||x|||_{\widetilde{A}} |||y|||_{\widetilde{A}},
\end{aligned}$$

then, one has

$$\begin{aligned}
\|xy\|_{\widetilde{A}} &= \max\{|||xy|||_{\widetilde{A}}, |\alpha\beta|\} \\
&\leq \max\{|||x|||_{\widetilde{A}} |||y|||_{\widetilde{A}}, |\alpha||\beta|\} \\
&\leq \max\{|||x|||_{\widetilde{A}} |||y|||_{\widetilde{A}}, |||x|||_{\widetilde{A}} |\beta|, |\alpha| |||y|||_{\widetilde{A}}, |\alpha||\beta|\} \\
&= \max\{|||x|||_{\widetilde{A}}, |\alpha|\} \max\{|||y|||_{\widetilde{A}}, |\beta|\} \\
&= \|x\|_{\widetilde{A}} \|y\|_{\widetilde{A}}.
\end{aligned}$$

Thus $\|xy\|_{\widetilde{A}} \leq \|x\|_{\widetilde{A}} \|y\|_{\widetilde{A}}$. So, $(\widetilde{A}, \|\cdot\|_{\widetilde{A}})$ is a Banach algebra.

Next, we show that $(\widetilde{A}, \|\cdot\|_{\widetilde{A}})$ is a C^*-algebra. For any $x = (a,\alpha), y = (b,\beta) \in \widetilde{A}$ and $\zeta \in \mathbb{C}$,

$$(x^*)^* = (a^*, \overline{\alpha})^* = (a, \alpha) = x,$$

Then $(x^*)^* = x$.

$$(x+y)^* = (a+b, \alpha+\beta)^* = ((a+b)^*, \overline{\alpha+\beta}) = (a^* + b^*, \overline{\alpha} + \overline{\beta}) = x^* + y^*,$$

Then $(x+y)^* = x^* + y^*$.

$$(\zeta x)^* = (\zeta a, \zeta\alpha)^* = ((\zeta a)^*, \overline{\zeta\alpha}) = (\overline{\zeta} a^*, \overline{\zeta}\overline{\alpha}) = \overline{\zeta}(a^*, \overline{\alpha}) = \overline{\zeta} x^*,$$

Then $(\zeta x)^* = \overline{\zeta} x^*$.

$$(xy)^* = (ab + \beta a + \alpha b, \alpha\beta)^* = ((ab + \beta a + \alpha b)^*, \overline{\alpha\beta}) = (b^* a^* + \overline{\beta} a^* + \overline{\alpha} b^*, \overline{\beta}\overline{\alpha}) = y^* x^*,$$

Then $(xy)^* = y^* x^*$. Since the C^*-condition $\|x^* x\|_{\widetilde{A}} = \|x\|^2_{\widetilde{A}}$ on a Banach *-algebra $\widetilde{A}$ is equivalent to $\|x^*x\|_{\widetilde{A}} \geq \|x^*\|_{\widetilde{A}}\|x\|_{\widetilde{A}}$ and

$$\begin{aligned}
|||x^*x|||_{\widetilde{A}} &= \sup_{\|c\|_A \leq 1} \|(a^*a + \overline{\alpha}a + \alpha a^*)c + |\alpha|^2 c\|_A \\
&= \sup_{c \neq 0, \|c\|_A \leq 1} \frac{\|c^*\|_A \|(a^*a + \overline{\alpha}a + \alpha a^*)c + |\alpha|^2 c\|_A}{\|c^*\|_A} \\
&\geq \sup_{c \neq 0, \|c\|_A \leq 1} \frac{\|c^*(a^*a + \overline{\alpha}a + \alpha a^*)c + |\alpha|^2 c^*c\|_A}{\|c^*\|_A} \\
&= \sup_{\|c\|_A \leq 1} \frac{\|(c^*a^* + \overline{\alpha}c^*)(ac + \alpha c)\|_A}{\|c^*\|_A} \\
&= \sup_{\|c\|_A \leq 1} \frac{\|(ac + \alpha c)^*(ac + \alpha c)\|_A}{\|c^*\|_A} \\
&= |||x^*|||_{\widetilde{A}} |||x|||_{\widetilde{A}},
\end{aligned}$$

then, one has

$$\begin{aligned}
\|x^*x\|_{\widetilde{A}} &= \|(a^*a + \overline{\alpha}a + \alpha a^*, \overline{\alpha}\alpha)\|_{\widetilde{A}} \\
&= \max\{|||x^*x|||_{\widetilde{A}}, |\overline{\alpha}\alpha|\} \\
&\geq \max\{|||x^*|||_{\widetilde{A}}|||x|||_{\widetilde{A}}, |\overline{\alpha}||\alpha|\} \\
&= \|x^*\|_{\widetilde{A}}\|x\|_{\widetilde{A}}.
\end{aligned}$$

Thus $(\widetilde{A}, \|\cdot\|_{\widetilde{A}})$ is a C^*-algebra with unit $1_{\widetilde{A}} = (0, 1)$.

If $\widetilde{A}$ is isomorphic to $A \oplus \mathbb{C}$ as C^*-algebras, then there exists a *-isomorphism $f\colon \widetilde{A} \to A \oplus \mathbb{C}$. For any $(a, \alpha) \in A \oplus \mathbb{C}$, there exists $x \in \widetilde{A}$ such that $f(x) = (a, \alpha)$. So,

$$\begin{aligned}
f(1_{\widetilde{A}})(a, \alpha) &= f(1_{\widetilde{A}})f(x) = f(1_{\widetilde{A}}x) = f(x) = (a, \alpha), \\
(a, \alpha)f(1_{\widetilde{A}}) &= f(x)f(1_{\widetilde{A}}) = f(x1_{\widetilde{A}}) = f(x) = (a, \alpha).
\end{aligned}$$

Then $f(1_{\widetilde{A}})$ is a unit of $\widetilde{A}$. Let $f(1_{\widetilde{A}}) = (b, \beta)$ then for any $(a, \alpha) \in A \oplus \mathbb{C}$,

$$\begin{aligned}
(a, \alpha) &= (a, \alpha)f(1_{\widetilde{A}}) = (a, \alpha)(b, \beta) = (ab, \alpha\beta), \\
(a, \alpha) &= f(1_{\widetilde{A}})(a, \alpha) = (b, \beta)(a, \alpha) = (ba, \beta\alpha).
\end{aligned}$$

Thus $a = ab = ba$ and $\alpha = \alpha\beta = \beta\alpha$. So, b is a unit in A. This implies A is a unital C^*-algebra. Conversely If A is a unital C^*-algebra with a unit 1_A, then $f\colon \widetilde{A} \to A \oplus \mathbb{C}$ defined by $f(a, \alpha) = (a + \alpha 1_A, \alpha)$ is a *-isomorphism. Indeed, $g\colon A \oplus \mathbb{C} \to \widetilde{A}$ defined by $g(a, \alpha) = (a - \alpha 1_A, \alpha)$ is a inverse map of f and

$$\begin{aligned}
f((a, \alpha)(b, \beta)) &= f(ab + \beta a + \alpha b, \alpha\beta) \\
&= (ab + \beta a + \alpha b + \alpha\beta 1_A, \alpha\beta) \\
&= ((a + \alpha 1_A)(b + \beta 1_A), \alpha\beta) \\
&= (a + \alpha 1_A, \alpha)(b + \beta 1_A, \beta) \\
&= f(a, \alpha)f(b, \beta),
\end{aligned}$$

and

$$
\begin{aligned}
f((a,\alpha)^*) &= f(a^*,\overline{\alpha}) \\
&= (a^* + \overline{\alpha}1_A, \overline{\alpha}) \\
&= ((a+\alpha 1_A)^*, \overline{\alpha}) \\
&= (a+\alpha 1_A, \alpha)^* \\
&= f(a,\alpha)^*.
\end{aligned}
$$

□

References

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