The proof for the five color theorem

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1 Introduction

There is a very famous theorem in graph theory called the four color theorem, which states that every loopless plane graph is 4-colorable. As a consequence of this theorem, every map can be colored with at most four colors so that no two adjacent regions have the same color. Although the four color theorem is known to be very difficult to prove, there is a weaken version of this theorem that can be proven much more easily:

Theorem 1.1 (Five Color Theorem)**.** *Every loopless plane graph is 5-colorable.*

The purpose of this article is to prove this theorem.

2 Auxiliary Lemma

We state an important lemma that we use in the proof of the five color theorem, which has already been proven in the lecture. Those who are familiar with the lecture notes can skip this section. First of all, the meaning of every terminology and notation used in this article is the same as in the lecture. However, let us just recall the definition of plane graphs for clarity.

Definition 2.1. *A* ***plane graph*** *is a finite graph* $G = (V, E)$ *with the set of vertices* V *given by* $\{x_1, x_2, \ldots, x_N\} \in \mathbb{R}^2$*, with the set of edges* E *given by a finite family of simple arcs (bijective and bicontinuous images of* $[0, 1]$*) having endpoints in* V*, and such that the interior of any arc contains no vertex and no point of any other edge.*

Note that this definition does not assume that plane graphs are simple. As written in the lecture notes, it depends on each author if plane graphs are assumed to be simple or not. However, it should be noted that when we discuss the coloring of loopless plane graphs, multiple edges do not play a role at all, and next lemma for a connected simple graph plays an important role in the proof of Theorem 1.1:

Lemma 2.2. *Let* $G = (V, E)$ *be a connected simple plane graph with* $|V| \geq 3$*. Then the following inequality holds:*

$$|E| \leq 3|V| - 6 \tag{1}$$

Observe that the assumption that G is simple cannot be eliminated. Indeed, if G is allowed not to be simple, then the inequality does not hold because we can choose two vertices and increase the number of edges between them arbitrarily without changing the number of vertices.

3 The proof for the main theorem

In this section, we prove Theorem 1.1.

Proof. (Proof for Theorem 1.1) Let $G = (V, E)$ be any loopless plane graph. First, we may assume that G is simple since multiple edges do not play a role at all in coloring. Since G is simple, we can use the notation $(x, y) \in V \times V$ for an edge e such that $i(e) = (x, y)$. Also, since G is a union of its connected components, it suffices to show that G is 5-colorable when G is connected. Hence we may assume that G is connected and simple. We give a proof by induction on $|V|$. If $|V| \leq 5$, then by assigning different colors to each vertex, G is 5-colorable. Suppose that the statement holds for $|V| = n$ for some integer $n \geq 5$. Let us show that the statement holds for $|V| = n + 1$. Let $|V| = n + 1$.
Claim 1 There exists $v \in V$ such that $\deg(v) \leq 5$.
Proof for Claim 1
Suppose for any $v \in V$, $\deg(v) \geq 6$. Since each vertex has at least 6 edges starting from it and one edge is shared by exactly two vertices, one has

$$6|V| \leq 2|E| \tag{2}$$

On the other hand, since G is a connected simple plane graph with $|V| \geq 3$, it follows from Lemma 2.2 that

$$|E| \leq 3|V| - 6 \tag{3}$$

Combining the equations (2) and (3), one has

$$6|V| \leq 6|V| - 12,$$

which is a contradiction. This proves Claim 1.

Then let v be a vertex of degree 5 or less, and let $H = G - \{v\}$. By the induction hypothesis, H is 5-colorable. Hence there exists a vertex 5-coloring $\omega : V \setminus \{v\} \to \{1, \ldots, 5\}$. If ω uses at most 4 colors for the neighbors of v, then we can color v by the color that is not used. Thus we let $\deg(v) = 5$ and assume that the neighbors of v have distinct colors for the rest of the proof.

Let D be an open small disk such that it meets only five edges starting from v and does not contain any other edges or vertices other than v. Let us label the intersection of those five edges with D according to their cyclic position in D as $s_1, \ldots, s_5$, and let (v, v_i) be the edge containing s_i. Without loss of generality, we may assume that $\omega(v_i) = i$ for each i. The purpose of taking D is to examine the behavior of our graph near v. See Figure 1.

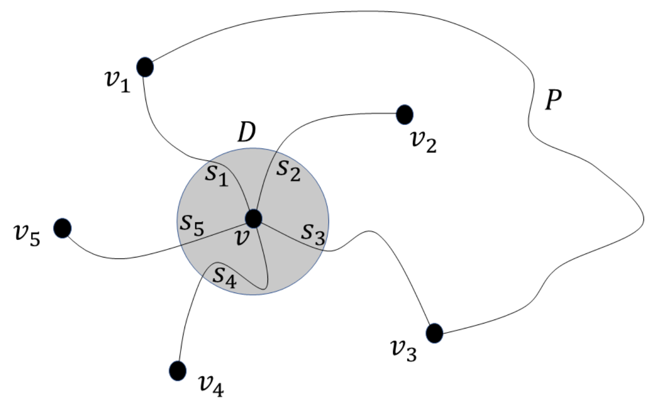

Figure 1:

Let P be any $\{v_1\} - \{v_3\}$ path in $H - \{v_2, v_4\}$.

Claim 2 If there exists a $\{v_1\} - \{v_3\}$ path P in $H - \{v_2, v_4\}$, then there does not exist a $\{v_2\} - \{v_4\}$ path in $H - P$. (We define the notation $H - P$ as follows: Let V_P be the set of all vertices contained in P. Define $H - P := H - V_P$.)

Proof for Claim 2

Let C be the cycle vv_1Pv_3v. (Since G is simple, the cycle C is uniquely determined.) It suffices to show that there does not exist a $\{v_2\} - \{v_4\}$ path in $G - C$. (The notation $G - C$ is defined to be $G - V_C$, where V_C is the set of all vertices contained in C.) Indeed, if there exists a $\{v_2\} - \{v_4\}$ path Q in $H - P$, then Q is a $\{v_2\} - \{v_4\}$ path in $G - C$. By taking a contraposition, if there does not exist a $\{v_2\} - \{v_4\}$ path in $G - C$, then there does not exist a $\{v_2\} - \{v_4\}$ path in $H - P$. Let $x_2 \in s_2$ and $x_4 \in s_4$. Since C is a cycle(, which implies C has no self-intersection as a closed path in $\mathbb{R}^2$ by the property of plane graphs), $\mathbb{R}^2 \setminus C$ has exactly two components: a bounded open set A and an unbounded open set B. (Strictly speaking, we use Jordan curve theorem here.) One has $x_2 \in A$ and $x_4 \in B$. Suppose there exists a $\{v_2\} - \{v_4\}$ path in $G - C$. By the property of plane graphs and $\{v_2, v_4\} \subseteq \mathbb{R}^2 \setminus C$, (v, v_2) and (v, v_4) contribute paths (in the topological sense) in $\mathbb{R}^2 \setminus C$ between x_2 and v_2 and between x_4 and v_4, respectively. Thus there exists a path in $\mathbb{R}^2 \setminus C$ between x_2 and x_4. It contradicts $x_2 \in A$ and $x_4 \in B$. This proves Claim 2.

Given $i, j \in \{1, \ldots, 5\}$, let $H_{i,j}$ be the subgraph of H induced by the vertices colored i or j.

Claim 3 We may assume that $H_{1,3}$ contains a $\{v_1\} - \{v_3\}$ path P in $H - \{v_2, v_4\}$.

Proof for Claim 3

If the component C_1 of $H_{1,3}$ containing v_1 also contains v_3, then Claim 3 holds. Suppose that the component C_1 of $H_{1,3}$ containing v_1 does not contain v_3. If we interchange the colors 1 and 3 at all the vertices of C_1, we obtain another 5-coloring of H. Then v_1 and v_3 are both colored 3 in this new coloring, and we may assign color 1 to v. Thus it suffices to deal with the case where the component C_1 of $H_{1,3}$ containing v_1 also contains v_3. This proves Claim 3.

Then the component C_2 of $H_{2,4}$ containing v_2 does not contain v_4. Indeed, if C_2 contains v_4, then C_2 is a $\{v_2\} - \{v_4\}$ path in $H - P$ since P is contained in $H_{1,3}$. This contradicts Claim 2. It means v_2 and v_4 lie in different components of $H_{2,4}$. If we interchange the colors 2 and 4 in C_2, we obtain a new 5-coloring of H. Since v_4 is not contained in C_2, v_2 and v_4 are colored 4 in this new coloring. Now v no longer has a neighbor colored 2. Thus we can assign color 2 to v. This completes the proof for Theorem 1.1.

□

Note that Claim 1, proven by Lemma 2.2, was one of the most crucial steps. By Claim 1, one can find a vertex v that has a degree of 5 and one may assume neighbors of v have distinct colors since one has 5 colors. If we could show that for any loopless plane graph there exists a vertex v that has a degree of 4 or less, then by applying the proof for Theorem 1.1, the four color theorem could be proven. However, it is known that there exists a loopless plane graph such that every vertex has a degree of 5 or more, called an icosahedral graph. Thus the proof for Theorem 1.1 cannot be used to prove the four color theorem.

References

[Die] R. Diestel, Graph theory, Fifth edition, Springer, 2017

[1] https://en.wikipedia.org/wiki/Five_color_theorem