On vertex and edge connectivity
Proof of inequality (5.1)

Quang Nhat Nguyen and Arata Suzuki

July 20, 2020

1 Proof of: $\kappa_E(G) \leq \delta(G)$

Let us assume that the minimum number of edges whose removal can disconnect $G(V, E)$ is:

$$\kappa_E(G) = k > \delta(G) \tag{1}$$

Consider a vertex $x_i \in V$ satisfying: $\deg(x_i) = \delta(G)$. If one removes all the edges $e \in E$ satisfying: $e = (x_j, x_i), x_j \in V$, the vertex x_i will then have degree 0 and thus is disconnected from G.

The number of such edges e mentioned above is: $k' = \deg(x_i) = \delta(G)$. This number is smaller than the number of k which was assumed to be the minimum and thus contradicts our initial assumption.

Therefore: $\kappa_E(G) \leq \delta(G) \quad \square$

2 Proof of: $\kappa_V(G) \leq \kappa_E(G)$

Let us assume that: $\kappa_V(G) > \kappa_E(G)$

Denote $E^* \subset E$ as the set of edges that correspond to the edge connectivity $\kappa_E(G)$. This means that if one denotes $\#(E^*)$ as the number of elements in E^* then: $\#(E^*) = \kappa_E(G)$.

Denote $V^* \subset V$ as the set of endpoints of all of edges in E^*.

Let G^* be a subgraph of G and defined by the sets V^* and E^*.

If we consider the case that between any two vertices in V^* there can be a maxmimum of one edge in E^*, then all possible values for the number of elements in V^*, denoted as

$\#(V^*)$, have to be within the range:

$$\kappa_E(G) \leq \#(V^*) \leq 2\kappa_E(G) \tag{2}$$

For any graph G^* in this case, there is always a choice of $\kappa_E(G)$ vertices such that they are the endpoints of all edges in E^*. If one removes these vertices from G^*, all of the edges in E^* will be removed as well. If this happens, the graph G will then become disconnected because the edges in E^* correspond to the edge connectivity $\kappa_E(G)$.

In case one allows more than one edge between two vertices, the minimum possible value of $\#(V^*)$ may become smaller than $\kappa_E(G)$. If it is, the number of vertices that are needed to be remove to make G disconnected will be equal to this minimum possible value, which is even smaller than $\kappa_E(G)$.

The above fact has proved that there is a choice of $\kappa_E(G)$ (or less) vertices whose removal can either disconnect G or reduce it to a 1-vertex graph. This contradicts the intitial assumption that $\kappa_V(G) > \kappa_E(G)$.

Therefore: $\kappa_V(G) \leq \kappa_E(G)$ $\square$