Graph Theory
Some problems of spanning trees

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Problem1
Prove that simple graph G is connected if and only if G has a spanning tree.

Proof. (Problem1)
If G has a subgraph G_1 as a spanning tree, then all vertices in G is connected by a path in the spanning tree G_1. Hence, G is connected.

If G is connected, let's consider subgraph G_1 that is connected and contains all the vertex and has the least number of edges. If G_1 has a cycle, then there exsists 2 vertices a and b that are connected by an edge and another path. Deleting that edge, all vertices are still connected since a and b are still connected. However, by deleting this edge connecting a and b, then,we obtain a new subgraph which is connected and contains all the vertices with one less edge than G_1 and therefore our initial G_1 is not the subgraph contains all the vertices and connected with the lease number of edges. By contradiction, G_1 does not have any cycle. Thus, G_1 is a spanning tree . □

Problem2
Prove that if a connected graph G is not a tree, then G has at least three spanning trees.

Proof. (problem2)
Since G is a connected graph but not a tree, then G has a cycle. From problem1, connected G has one spanning tree T_1. Let be the graph of the cycle be C. Since the tree does not have any sycle, there exsists an edge of C that does not belong to T_1. Let that edge e connecting endpoints x and y. Spanning tree includes x and y. Thus, there exsists the path D in T_1 connects x and y in the spanning tree which does not include edge e.
Now T_2 is equivalent to T_1 deleting an edge d_1 on the path D and connecting x and y by e.
T_3 is equivalent to T_1 deleting an edge d_2 on the path D and connecting x and y by e.
We can choose d_1 and d_2 different from each other since the path connecting x and y in T_1 has the walk of length at least 2. If not, the graph will have multiple edges, then the graph is not a tree.

Therefore By this construction, the connected graph G(not a tree) has at least 3 spanning trees. □

Problem3

Let v be a vertex in a connected graph G. Prove that there exsists a spanning tree T of graph G such that distances to every vertex from v are the same in G and in T.

Proof. (problem3)
Let the graph be graph G=(V, E)

1.We construct the graph G_1 containing the path from v to all vertices with the following constraiants:
- First, we construct the shortest path of length 1 from v to all vertices in its neighborhood. Since we are considering simple graph (because of (problem1)), each path is unique.
- We construct the shortest path between v and x_i whose lenghth is greater or eaqual to 2 with the following constraiants:

If this shortest path between v and x_i includes the edge $x_j x_i$ connection x_j to x_i, then this shortest path is constructed by the shortest path from v to x_j and the edge $x_j x_i$ (we do this in order to make sure there is no cycle that is constructed by two shortest path of the same shortest length from v to x_j). We now prove that the path between v to x_i is unique by induction:
We have x_i that are neighborhood of v, then the paths are unique and of lenghth 1 due to the first point above.

Assume that the shortes path from v to x_j is constructed to be unique. Then for any vertex x_i such that the shortest path between v and x_i contains x_j , the shortest path from v to x_j is unique since the path between v and x_i is unique, and the edge between x_j and x_i is also unique
By induction, the shortest path between v and any vertex x_i of graph G is unique.
- As we constructed G_1 includes all the shortest path from v to other vertices, G_1 contains all the vertex in V. Let G_1=(V, E_1) where $E_1$is the set of the edges in all of these shortest paths we constructed.
- By this construction, the distance between v and each other vertex is the same in G_1 and in G_1.

2.Let us prove that G_1 is a spanning tree:
- G_1 is connected since for vertices x and y different from v, x is connected to v and y is connected to v, thus x and y are connected.
- G_1 contains all the vertices in the graph.
- Suppose there is a cycle C in G_1. Consider a vertex x_1 in the cycle C, then there are 2 different path from v to x_1. This is a contradiction to our construction and uniquness of the shortest path from v to each vertex. Therefore, by contradiction, G_1 is acyclic.

-Therefore,G_1 is aspanning tree of G that satisfies the distance between v and any vertex is the same in G and in G_1.

□