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Lemma 8.3: Let e be a cross-edge of a dfs tree performed on a directed graph. If the origin of e is x and the terminal vertex of e is y, with x and y belonging to the tree, then one has $dfnumber(x) > dfnumber(y)$.

Proof:

We are going to prove by contradiction. Assume that there exists a cross-edge $e = (x, y)$ such that $dfnumber(x) < dfnumber(y)$. After conducting DFS, we obtained the dfs-tree, denoted by T. The cross-edge $e = (x, y)$ implies that x and y are not in the same "family", in other words, neither of them is the ancestor of the other one. This means that there exists a subtree T_x of T which contains x but not y and a subtree T_y of T which contains y but not x , together with a vertex a in T being the root of the minimal subtree of T containing both T_x and T_y. Among all the roots corresponding to subtrees of T that contains both T_x and T_y, the vertex a has the largest depth.

Based on the rules of DFS, since $dfnumber(x) < dfnumber(y)$, one has to finish discovering all vertices in T_x, then backtrack to a, before proceeding to T_y. However, the existence of the edge e, which points from x to y, means that x is not finished yet, which contradicts to the operation of DFS. Hence, the assumption is not correct and thus $dfnumber(x) > dfnumber(y)$.

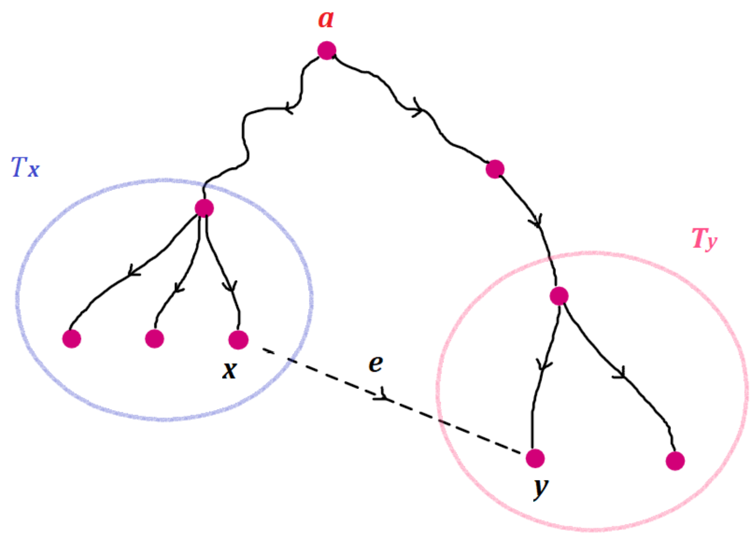

Figure 1: Subtree of T with its root a and both T_x and T_y contained