Exercise 2.1.10

$a \in C$: invertible $b \in C$ suchas $\|b-a\| < \|a^{-1}\|^{-1}$
then b : invertible and $\|b^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a-b\|$ and $a \sim_h b$ in $GL(C)$

$\because$)

$\|I - a^{-1}b\| = \|a^{-1}(a-b)\| \leq \|a^{-1}\| \cdot \|a-b\| < 1$

by Neumann series $a^{-1}b$ is invertible.

hence $b^{-1} = (a^{-1}b)^{-1}a^{-1}$, b : invertible

since $\|b^{-1}\| \leq \|(a^{-1}b)^{-1}\| \|a^{-1}\|$

$\|b^{-1}\|^{-1} \geq \|(a^{-1}b)^{-1}\|^{-1} \|a^{-1}\|^{-1}$

(also by Neumann series

$\|(a^{-1}b)^{-1}\| \leq 1 + \|I - a^{-1}b\|^2 + \|I - a^{-1}b\|^3 + \dots$
$= (1 - \|I - a^{-1}b\|)^{-1}$)

$\geq (1 - \|I - a^{-1}b\|) \|a^{-1}\|^{-1}$

$= \|a^{-1}\|^{-1} - \|a^{-1}\|^{-1} \|I - a^{-1}b\|$

($\|I - a^{-1}b\| = \|a^{-1}(a-b)\| \leq \|a^{-1}\| \|a-b\|$
$\|a^{-1}\|^{-1} \|I - a^{-1}b\| \leq \|a-b\|$)

$\geq \|a^{-1}\|^{-1} - \|a-b\|$

denote $C_t = (1-t)a + tb$ $(t \in [0,1])$

$\|a - C_t\| = t\|a-b\| < \|a^{-1}\|^{-1}$

therefore C_t : invertible and clealy C_t : continuous

$a = C_0$, $b = C_1$ so $a \sim_h b$ in $GL(C)$.

Exercise 2.2.2

C: C^* alg $p \in P(C)$ then $\sigma(p) \subset \{0,1\}$

$\because$) by def of projection

$p - p^2 = 0.$

by spectral mapping thm if $\lambda \in \sigma(p)$ then

$\lambda - \lambda^2 = 0 \Leftrightarrow \lambda = 0$ or $\lambda = 1$. so $\sigma(p) \subset \{0,1\}$.

Exercise 2.2.3.

Show Murray-v.N equivarence is transitive.

$\because$) $p, q, r \in P(C)$

$p \sim q$, $q \sim r$ ($\sim$: Murray-v.N equivalence)

by $p \sim q$, $\exists\, u \in C$ s.t $u^*u = p$, $uu^* = q$

also $\exists\, v \in C$ s.t. $v^*v = q$, $vv^* = r$

denote $w = vu$

$w^*w = u^*v^*vu = u^*qu = u^*up = p^2 = p.$

$ww^* = vuu^*v^* = vqv^* = rvv^* = r^2 = r.$

so $p \sim r$