Exercise 2.1.6

(i) prove $\mathcal{U}_0(C) \triangleleft \mathcal{U}(C)$

first $\mathcal{U}_0(C)$ is closed wrt group multiplication.

Indeed $u_1, u_2 \in \mathcal{U}_0(C)$

$\exists\ w_1(t) : [0,1] \to \mathcal{U}(C)\quad w_1(0) = u_1,\ w_2(1) = 1$

$\exists\ w_2(t) : [0,1] \to \mathcal{U}(C)\quad w_2(0) = u_2,\ w_2(1) = 1$

$u_1 u_2 \sim_h 1$ by $w_1 w_2(t)$. $u_1 u_2 \in \mathcal{U}_0(C)$.

Next $\mathcal{U}_0(C)$ is closed wrt inverse.

Indeed $u \in \mathcal{U}_0(C)$

$\exists\ w(t) : [0,1] \to \mathcal{U}(C)\quad w(0) = u\ w(1) = 1$

Now we can think $w^{-1}(t)$ because $w(t) \in \mathcal{U}(C)$.

and unitary ops are invertible,

$w^{-1}(t)$ is a path from u^{-1} to 1

last, $vuv^* \in \mathcal{U}_0(C)\quad v \in \mathcal{U}(C)\ u \in \mathcal{U}_0(C)$.

$\exists\ w(t) : [0,1] \to \mathcal{U}(C)\quad w(0) = u\ w(1) = 1$.

$vuv^* \sim_h v^* v = 1\quad (v \in \mathcal{U}(C)$ $\square$

(ii) prove $\mathcal{U}_0(C)$ is open and closed relative to $\mathcal{U}(C)$.

$G := \{\exp(ih_1) \cdots \exp(ih_n) \mid n \in \mathbb{N}\ h_1, \cdots, h_n : \text{self-adj}\}$

by lemma 2.1.2 (i) and $\mathcal{U}_0(C)$ is group (closed wrt multiplication)

so $G \subset \mathcal{U}_0(C)$ and G is also a group. (easy to see)

Now we will show $G \subset \mathcal{U}(C)$ is open.

indeed $v \in G$ and take $u \in \mathcal{U}(C)$ s.t. $\|u - v\| < 2$

by lemma 2.1.2 (iii) and $\|1 - uv^*\| = \|u - v\| < 2$. $1 \sim_h uv^*$

and $-1 \notin sp(uv^*)$.

in the proof of lemma 2.1.2 (ii) $\exists h$: self-adj $uv^* = e^{ih}$

so

$u = \exp(ih)v \in G$.

this implies G is open.

next we will prove G is closed

by decomposition of coset

$$\mathcal{U}(C) = \underset{\substack{\| \\ G}}{Ge} \bigsqcup_{\substack{u \in \mathcal{U}(C) \\ u \neq e}} Gu$$

So.

$$\mathcal{U}(C) \setminus G = \bigsqcup_{\substack{u \in \mathcal{U}(C) \\ u \neq e}} Gu$$

$G \overset{\text{hom}}{\simeq} Gu$ (indeed u is invertible).

and we've already known G: open

so $\mathcal{U}(C) \setminus G$: is open hence G: closed. $\square$

(iii) prove $\mathcal{U}(C) = G$.

G is a nonempty subset of $\mathcal{U}_0(C)$, G: open closed

$\mathcal{U}_0(C)$ is connected (this is clear by def of $\mathcal{U}_0(C)$).

$\mathcal{U}_0(C) = G$.

$W^{-1}(t)$: Continuous.

fix $a \in (0,1)$

$$\| W^{-1}(a+t) - W^{-1}(a) \| = \| W^{-1}(a+t) (W(a) - W(a+t)) W^{-1}(a) \|$$

$$\leq \| W^{-1}(a+t) \| \| W(a) - W(a+t) \| \| W^{-1}(a) \|$$

$$= \| W(a) - W(a+t) \| \quad (\because \|U\| = 1, \ U : \text{unitary})$$

Since $W(t)$ is cont. $W^{-1}(t)$ is cont

$W'(t) = U W(t) U^*$: Cont

$$\| W'(a+t) - W'(t) \| \leq \| U \| \| W(a+t) - W(t) \| \| U^* \|$$

$$= \| W(a+t) - W(t) \|$$