Exercise 1.1.6 $\varphi : C \to Q$: $*$-homomorphism

(i) φ : isometric $\Leftrightarrow$ φ : injective

($\Rightarrow$) if $a \neq b \Rightarrow \|a-b\| > 0$

so $\|\varphi(a) - \varphi(b)\| = \|\varphi(a-b)\| = \|a-b\| > 0$

hence $\varphi(a) \neq \varphi(b)$ injective.

($\Leftarrow$) by C^*-condition $\|x^*x\| = \|x\|^2$.

since x^*x is normal, denote $r(x)$ is spectral radius.

$$\|x\|^2 = \|x^*x\| = r(x^*x)$$

and $\sigma(\varphi(x)) \subseteq \sigma(x)$

$$\|x\|^2 = r(x^*x) \geq r(\varphi(x^*x)) = \|\varphi(x)\|^2$$

. φ is injective so $\exists \varphi^{-1} : (\mathrm{Im}\,\varphi) \to C$ and φ^{-1} is also C^*-hom

$$\|x\|^2 \leq \|\varphi(x)\|$$

in this proof I assumed C^*-alg are unital but considering the minimal unitizations this proof extends in general.

(ii) Prove $\mathrm{Ker}(\varphi)$ is a sub C^*alg of C, $\mathrm{Ran}(\varphi)$ is so of Q

by $\|\varphi(x)\|^2 \leq \|x\|^2$ φ is continuous.

so Kernel is closed $x_n \to x$ $x_n \in \mathrm{Ker}(\varphi)$ $\varphi(x) = \lim \varphi(x_n) = 0_Q$

~~other condition is goes well.~~

$\mathrm{Ran}(\varphi)$ is closed because

$\bar{\varphi} : C/\mathrm{Ker}(\varphi) \longrightarrow \mathrm{Ran}(\varphi)$ is injective so isometric

$\bar{\varphi}(x + \mathrm{Ker}(\varphi)) = \varphi(x)$

and $C/\mathrm{Ker}(\varphi)$ is complete so $\mathrm{Ran}(\varphi)$ is closed.

We've already shown $Ker(\varphi)$, $Ran(\varphi)$ are norm-closed.

Next. we will show $Ker(\varphi)$, $Ran(\varphi)$ is sub alg.

By a general consideration of linear alg.

$Ker(\varphi)$, $Ran(\varphi)$ are sub linear sp.

also $x, y \in Ker(\varphi)$ $\varphi(xy) = \varphi(x)\varphi(y) = 0$ $xy \in Ker(\varphi)$

$\varphi(x^*) = \varphi(x)^* = 0$ $x^* \in Ker(\varphi)$

$a, b \in Ran(\varphi)$ $\varphi(x) = a$, $\varphi(y) = b$ $ab = \varphi(x)\varphi(y) = \varphi(xy)$ $ab \in Ran(\varphi)$

$a^* = \varphi(x)^* = \varphi(x^*)$ $a^* \in Ran(\varphi)$

So $Ker(\varphi)$, $Ran(\varphi)$ are closed wrt multiplication and involution

Hence $Ker(\varphi)$, $Ran(\varphi)$ are sub C^*-alg.