Exercise 3.1.4

(ii) $\Leftrightarrow$ (iii) It suffices to show $tr(p) = rank(p)$ for $p \in \mathcal{P}(M_n(\mathbb{C}))$, where rank denotes the usual rank of matrices. Since p is a projection, we have $p(p - I_n) = 0$. This implies that the minimal polynomial of p is $X(X-1)$. Hence, it follows that p can be diagonalised into $diag(1, \cdots, 1, 0, \cdots, 0)$. Thus we conclude

$$tr(p) = (\text{multiplicity of the eigenvalue } 1) = rank(p).$$

(i) $\Rightarrow$ (iii) Suppose $p \sim q$. Then, by definition, there exists $v \in M_n(\mathbb{C})$ such that

$$p = v^* v \quad \text{and} \quad q = v v^*.$$

This implies $tr(p) = tr(q)$.

(iii) $\Rightarrow$ (i) Since we suppose $p, q \in \mathcal{P}(M_n(\mathbb{C}))$, both p and q are Hermitian matrices. Hence, under the assumption (iii), there exist unitary matrices A and B such that

$$A^{-1} p A = B^{-1} q B = diag(1, \cdots, 1, 0, \cdots, 0).$$

Let us denote $diag(1, \cdots, 1, 0, \cdots, 0)$ as M.

Then, we have

$$(qBA^*p)^*(qBA^*p)$$

$$= p^*AB^*q^*qBA^*p$$

$$= p^*AB^{-1}qBA^{-1}p \quad \left(\because \begin{array}{l} q \in \mathcal{P}(M_n(\mathbb{C})) \\ A \text{ and } B \text{ are unitary} \end{array} \right)$$

$$= p^*AMA^{-1}p$$

$$= p^*p^2$$

$$= p \quad (\because p \in \mathcal{P}(M_n(\mathbb{C}))).$$

Moreover, we have

$$(qBA^*p)(qBA^*p)^*$$

$$= qBA^*pp^*AB^*q^*$$

$$= qBA^{-1}pAB^{-1}q^* \quad \left(\because \begin{array}{l} p \in \mathcal{P}(M_n(\mathbb{C})) \\ A \text{ and } B \text{ are unitary} \end{array} \right)$$

$$= qBMB^{-1}q^*$$

$$= q^2q^*$$

$$= q \quad (\because q \in \mathcal{P}(M_n(\mathbb{C}))).$$

Hence we conclude $p \sim q$.

$\square$

We show $\mathcal{D}(\mathbb{C}) \simeq \mathbb{Z}_+$.

Consider $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$ with $m \leq n$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$. If $m = n$, then $[A]_{\mathcal{D}} = [B]_{\mathcal{D}}$.

If $m < n$, then we have $A \sim_0 A \oplus O_{n-m} \sim_0 B$.

[$\because$ The first equivalence follows from Prop 3.1.1 (i).
The second is clear in view of the definition of "$\oplus$" and trace.

Hence, in $\mathcal{D}(\mathbb{C})$, we identify matrices with the same trace regardless of the sizes of the matrices, that is,

$$[A]_{\mathcal{D}} = \{ B \in \mathcal{P}_\infty(\mathbb{C}) \mid \mathrm{tr}(B) = \mathrm{tr}(A) \}.$$

We define a map $\varphi : \mathcal{D}(\mathbb{C}) \longrightarrow \mathbb{Z}_+$ by

$$\varphi([A]_{\mathcal{D}}) = \mathrm{tr}(A).$$

This is obviously well-defined and bijective.

We have to verify that φ is a semigroup homomorphism.

We have

$$\begin{aligned} \varphi([A]_{\mathcal{D}} + [B]_{\mathcal{D}}) &= \mathrm{tr}([A \oplus B]_{\mathcal{D}}) && \left(\because \text{ by the definition of } \varphi \text{ and "+"}\right) \\ &= \mathrm{tr}(A) + \mathrm{tr}(B) && \left(\because \text{ by the definition of "}\oplus\text{" and trace}\right) \\ &= \varphi([A]_{\mathcal{D}}) + \varphi([B]_{\mathcal{D}}), \end{aligned}$$

which completes the proof. $\square$