Exercise 2.2.2 Let $\mathcal{C}$ be a unital C^*-algebra and let p be a projection in $\mathcal{C}$. Then, we have

$$\sigma(p) \subset \{0, 1\}.$$

(Proof) Clearly we have $\sigma(\mathbb{0}) = \{0\}$ and $\sigma(\mathbb{1}) = 1$.

Hence we consider $p \neq 0, \mathbb{1}$.

Since p is a projection, we have $p(\mathbb{1} - p) = \mathbb{0}$, which implies $\{0, 1\} \subset \sigma(p)$. Conversely, for $\lambda \neq 0, 1$, we have

$$(p - \lambda \mathbb{1}) \left\{ \frac{1}{\lambda(1-\lambda)} p - \frac{1}{\lambda} \mathbb{1} \right\}$$

$$= \frac{1}{\lambda(1-\lambda)} p^2 - \frac{1}{\lambda} p - \frac{1}{1-\lambda} p + \mathbb{1}$$

$$= \left\{ \frac{1}{\lambda(1-\lambda)} - \frac{1}{\lambda} - \frac{1}{1-\lambda} \right\} p + \mathbb{1} \quad (\because p \text{ is a projection})$$

$$= \mathbb{1}$$

and hence similarly

$$\left\{ \frac{1}{\lambda(1-\lambda)} p - \frac{1}{\lambda} \mathbb{1} \right\} (p - \lambda \mathbb{1}) = \mathbb{1}.$$

This implies $\sigma(p) = \{0, 1\}$. Thus we conclude

$$\sigma(p) = \begin{cases} \{0\} & \text{if } p = 0 \\ \{1\} & \text{if } p = 1 \\ \{0, 1\} & \text{else} \end{cases}.$$

$\square$