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P.7 Exercise 1.1.6

$\mathcal{C}, \mathcal{Q}$: C^*-algebra, $\varphi : \mathcal{C} \to \mathcal{Q}$: $*$-homomorphism,

(i) φ : isometric $\iff$ φ : injective

(ii) $\mathrm{Ker}\,\varphi$: C^* subalgebra of $\mathcal{C}$,

$\mathrm{Ran}\,\varphi$: C^*-subalgebra of $\mathcal{Q}$

Proof

(i) $\Rightarrow$) Let φ be isometric.

Then, if $\varphi(a) = \varphi(b)$ for $a, b \in \mathcal{C}$,

$$\begin{aligned} \|a-b\| &= \|\varphi(a-b)\| && (\because \varphi : \text{isometric}) \\ &= \|\varphi(a)-\varphi(b)\| && (\because \varphi : \text{homomorphism}) \\ &= 0 && (\because \varphi(a) = \varphi(b)) \end{aligned}$$

Hence, $a = b$.

$\Leftarrow$) Let φ is injective.

Let $\psi := \varphi^{-1} : \mathrm{Im}\,\varphi \to \mathcal{C}$.

$\forall a \in \mathcal{C}$,

$$\begin{aligned} \|\varphi(a)\|^2 &= \|\varphi(a)^*\varphi(a)\| && (\because \mathcal{Q} : C^*\text{algebra}) \\ &= \|\varphi(a^*a)\| && (\because \varphi : *\text{-homomorphism}) \\ &= r(\varphi(a^*a)) && (\because a^*a : \text{hermitian}) \\ &\le r(a^*a) && (\because \sigma(\varphi(a^*a)) \subset \sigma(a^*a)) \end{aligned}$$

$= \|a^*a\|$ $\quad$ ($\because$ a^*a hermetian)

$= \|a\|^2$ $\quad$ ($\because$ $\mathcal{C}$ C^*-algebra)

So, $\|\varphi(a)\| \leq \|a\|$

Similarly,

$$\|a\| = \|\psi(\varphi(a))\| \leq \|\varphi(a)\|$$

Hence, $\|\varphi(a)\| = \|a\|$

(ii) · $\mathrm{Ker}\,\varphi$: C^* subalgebra of $\mathcal{C}$:

The only thing that needs to be proved is that $\mathrm{Ker}\,\varphi$ is closed.

Let $a_n \in \mathrm{Ker}\,\varphi$ $(n \in \mathbb{N})$, $a \in \mathcal{C}$ be $a_n \to a$.

Then $\forall n \in \mathbb{N}$, $\varphi(a_n) = 0$.

Since φ is continuous,

$$\varphi(a) = \lim_{n \to \infty} \varphi(a_n) = 0$$

Hence, $a \in \mathrm{Ker}\,\varphi$. i.e. $\mathrm{Ker}\,\varphi$ is closed.

· $\mathrm{Ran}\,\varphi$: C^*-subalgebra of $\mathcal{Q}$:

Since $\mathrm{Ker}\,\varphi$ is a closed ideal of $\mathcal{C}$,

$\mathcal{C}/\mathrm{Ker}\,\varphi$ is a C^*-algebra.

$$\left(\|a + \mathrm{Ker}\,\varphi\| := \inf_{b \in \mathrm{Ker}\,\varphi} \|a + b\|, \quad (a + \mathrm{Ker}\,\varphi)^* = a^* + \mathrm{Ker}\,\varphi \quad (a \in \mathcal{C}) \right)$$

By a homomorphism theorem,

$$\mathcal{C}/\mathrm{Ker}\,\varphi \cong \mathrm{Ran}\,\varphi$$

Hence, $\operatorname{Ran}\varphi$ is a C^* subalgebra of Q. $\square$