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Using Euler's theorem in 3-dimensions and proof of the number of regular polyhedrons

I will prove two things:

1. Any polyhedron can be transformed into a plane graph,
2. The reason why the number of regular polyhedrons is 5 by using Euler's theorem.

1. Any polyhedron can be transformed into a plane graph

It needs the concept of homeomorphic.

Definition:

If two topological spaces A, B admit a mapping $f\colon A \to B$ which is continuous and bijective, and $f^{-1}\colon B \to A$ is continuous, then A and B are homeomorphic.

It means "A can be deformed into B". If any polyhedron and a plane graph are homeomorphic, it means that any polyhedron can be deformed into a plane graph and one can use Euler's theorem. Also, it is a transitive relation. It means

(A and B are homeomorphic) *and* (B and C are homeomorphic)⇒(A and C are homeomorphic).

It is clear that any 2-polyhedron and the 2-sphere are homeomorphic, so if the 2-sphere (without one point) and a 2-dimensional plane are homeomorphic, any 2-polyhedrons and a 2-dimentional plane are homeomorphic.

Proof. Consider $S^2 = \{(x,y,z)|x^2+y^2+(z-1)^2=1,\ 0 \le z \le 2\}$, $p_0 = \{(0,0,2)\}$, and let $p=(x,y,z) \in S^2 - p_0$. One sets $f\colon S^2 - p_0 \to R^2$ by

$$f(p) = (\tfrac{2x}{2-z}, \tfrac{2y}{2-z}, 0).$$

Then, f is bijective and continuous, and for $p=(x,y,0)$ one has

$$f^{-1}(p) = \left(\frac{4x}{x^2+y^2+4}, \frac{4y}{x^2+y^2+4}, \frac{2x^2+2y^2}{x^2+y^2+4}\right)$$

Which is also continuous.

Clearly, one can chose p_0 everywhere. (expect at a vertex or on an edge). In this way, every polyhedron can be deformed in a plane graph.

2. The reason why the number of regular polyhedrons is 5

Consider ・a regular m-gon ・the number of vertex: V ・the number of edge: E ・the number of faces: F ・the degree: N

It is clear $m > 2$, $N > 2$.

・$mF = 2E \quad \therefore F = \frac{2E}{m}$ ・$NV = 2E \quad \therefore V = \frac{2E}{N}$ (handshaking lemma)

Input these numbers to Euler's theorem ($V - E + F = 2$)

$$V - E + F = \frac{2E}{N} - E + \frac{2E}{m} = 2$$

$$\frac{2}{m} + \frac{2}{N} - 1 = \frac{2}{E} > 0 \quad (E > 0)$$

$$\therefore \frac{2}{m} + \frac{2}{N} > 1 \qquad mN - 2m - 2N < 0 \qquad (m-2)(N-2) < 4$$

One can list the possibilities:

$$\therefore (m, N) = (3,3)(3,4)(3,5)(4,3)(5,3)$$

- $(m, N) = (3,3)$…regular tetrahedron
- $(3,4)$…regular octahedron
- $(3,5)$…regular icosahedron
- $(4,3)$…regular hexahedron
- $(5,3)$…regular dodecahedron

That is why, the number of regular polyhedrons is 5.