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GRAPH - RECONSTRUCTION PROBLEM

I. Definition: Vertex-deletion subgraph list

Let G be a graph with $V_G = \{v_1, v_2, \dots, v_n\}$
The vertex-deletion subgraph list of G is the list of n subgraphs $H_1, \dots, H_n$, where $H_k = G - v_k$ $(k = 1, 2, \dots, n)$

II. Definition: Reconstruction deck

The reconstruction deck of a graph is its vertex-deletion subgraph list, with no labels on the vertices. Each individual vertex-deletion subgraph is regarded as a ***card*** in the deck.

III. Definition: The graph-construction problem

The graph-reconstruction problem is to decide whether two non-isomorphic simple graphs with three or more vertices can have the same reconstruction deck.

IV. Example 1:

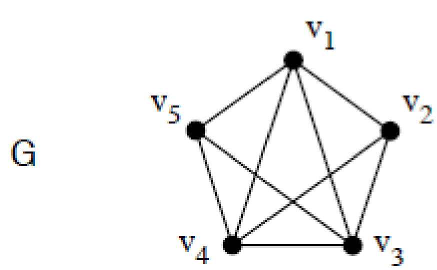


Figure 1: Graph G

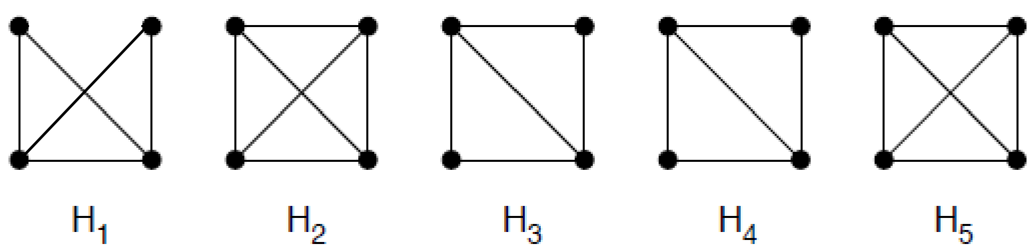


Figure 2: A graph G and its vertex-deletion subgraph list

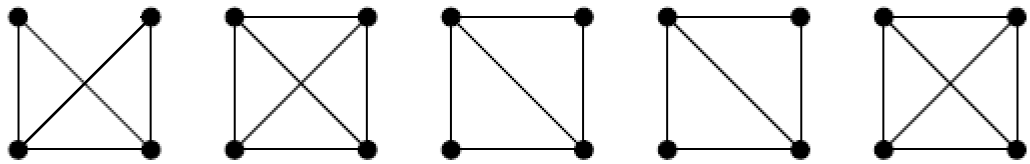


Figure 3: Reconstruction deck for the graph G

The graph-reconstruction problem asks whether graph G (figure 1) is the only graph (up to isomorphism) that has the deck shown in figure 3.

V. Reconstruction conjecture

Let G and G' be two graphs with at least three vertices and with $V_G = \{v_1, v_2, \ldots, v_n\}$ and $V_{G'} = \{w_1, w_2, \ldots, w_n\}$, such that $G - v_i \cong G' - w_i$, $i = 1, \ldots, n$. Then $G \cong G'$.

VI. Theorem

The number of vertices and the number of edges of a simple graph can be calculated from its vertex-deletion subgraph list.

Proof

- If there are n subgraphs in the vertex-deletion subgraph list, then number of vertices in the graph $|G| = n$
- Since each edge only appears in $n - 2$ subgraphs that do not contain either of its endpoints, the total number of edges of all subgraphs count n-2 times the number of edges of the original graph G. That is $||G|| = \frac{1}{n-2}\sum_{i=1}^{n}||G - v_i||$

VII. Corollary

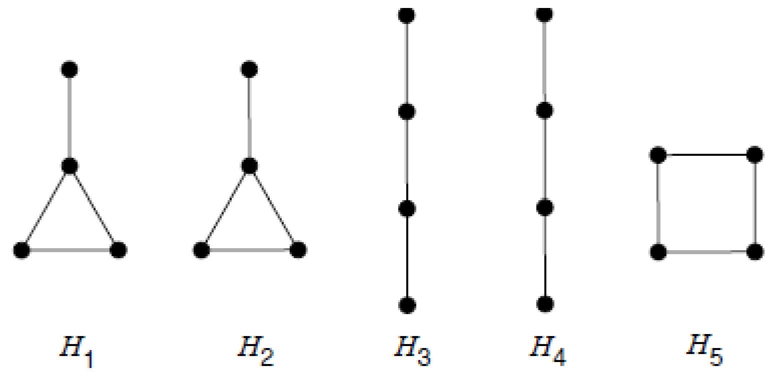
The degree sequence of a graph G can be calculated from its reconstruction deck

Proof:

First calculate the number of edges in G, $||G||$. For a card H_k, the degree of the missing vertex v_k is the difference between $||G||$ and the number of edges on that card, that is $deg(v_k) = ||G|| - ||H_k||$

VIII. Example 2

This example illustrates how to find a graph G given its reconstruction deck
Suppose G is a graph with the following reconstruction deck



|G| = 5

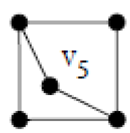
$||G|| = \frac{1}{|G| - 2} \sum_{k=1}^{5} ||H_k|| = 1/(5-2) \times (4+4+3+3+4) = 6$

Consider subgraph H_5

$Deg(v_5) = ||G|| - ||H_5|| = 6 – 4 = 2$

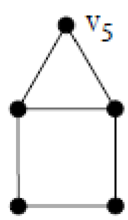
Hence, there are two ways in which v_5 could be joined to H_5

- Case 1: v_5 can be adjacent to two non-adjacent vertices of H_5.



Case 1

- Case 2: v_5 can be adjacent to two adjacent vertices.



Case 2

On trying to delete other vertices, we observe that only case 2 satisfy the whole reconstruction deck.

IX. Corollary

A regular graph can be reconstructed from its reconstruction deck

* A regular graph: a graph whose vertices all have equal degree. A k-regular graph í a regular graph whose common degree is k.

Proof:

Suppose G is a d-regular graph, for some d being the common degree of all vertices. From the reconstruction deck, we can determine the value of d. Each card in the deck would have d vertices, namely $|G - v_i| = \deg(v_i) = d$ for all i; and each vertex within a card would have d-1 degree. To reconstruct the graph, join those vertices to the missing vertex.

Reference: J.L. Gross, J. Yellen, M. Anderson, Graph theory and its applications, CRC press.