Nguyen Minh Diep – 061801864

Special mathematics lectures – Spring 2020

Proposition 3.2

Let G be a graph with n vertices. The following statements are equivalent:

i) G is a tree,

ii) G contains no undirected cycle and has n-1 edges,

iii) G is connected and has n-1 edges

iv) G is connected and every edge is a cut-edge

v) Any two vertices of G are connected by exactly one unoriented path (when the orientation on the edges is disregarded)

vi) G contains no undirected cycle, and the addition of any new edge e on the graph generates a graph with exactly one undirected cycle.

Proof:

If n = 1, all six statements are trivially true. Assume n ≥ 2

1. (i → ii)
   By definition, a tree is a connected graph with no underlying undirected cycle.
   Now we need to prove that a tree with n vertices contains exactly n – 1 edges.
   - n=2: A tree with 2 vertices has 1 edge connecting those 2 vertices
   - Induction hypothesis: Suppose a tree with n=k vertices has k-1 edges
     Consider a tree G on k+1 vertices. A tree G has one leaf v. The graph G-v is a tree with k vertices (since G-v is still connected and deleting one leaf does not create any undirected cycle). Therefore, by induction hypothesis, G-v has k-1 edges. Moreover, G has 1 edge more than G-v. Therefore, T has k edges.

2. (ii → iii)
   *Component of graph is a maximal connected subgraph (# components := c(G))
   A graph G with no undirected cycle is a forest. The forest G has n vertices
   Suppose G has k components. G_i with n_i vertices (i: 1→k), then$\sum_1^k n_i$ = n. Every component of G (G_i), connected and with no undirected cycle, is hence a tree. Therefore, every tree G_i has n_i-1 vertices. Hence, the total number of edges of graph G is n – c(G) = n – k.
   From ii), G has n-1 edges => k = 1
   Graph G with n vertices has n-1 components → G is connected.

3. (iii → iv)
   Let e be an edge of G
   From the assumption, G has 1 component.
   G has n-1 edge → G’ = G – e has n-2 edges
   G has n vertices → G’ has n vertices as well → G’ has at least n – c(G’) edges

(Suppose graph G' has k components that are connected. Each connected component G_i with n_i vertices has at least $n_i - 1$ edges → k components have at least n – k edges)
→ n-2 ≥ n – c(G') → c(G') ≤ 2
→ G – e has at least 2 components, which is at least 1 more component than G.
→ e is a cut-edge, which holds true for all e belonging to E of G.

4. (iv → v)
From iv, G with n vertices is connected and every edge is a cut-edge.
By contradiction, suppose x and y be two vertices that are connected by two different unoriented paths between them, P_1 and P_2. Let u is the first vertex from which the two paths diverge (u might be x); let v be the first vertex at which the path meets again. Then there is a undirected cycle formed from the path taken between u and v. Any edge on this cycle is not a cut-edge since the removal of such an edge does not disconnect the graph. This goes against iv. Then the hypothesis is wrong.

5. (v → vi)
Firstly, G contains no undirected cycle since any 2 vertices of G are connected by exactly one undirected path, so there is no undirected cycle between those edges.
Also, the addition of any new edge e (regardless of the orientation) to G will create a cycle, since each of 2 endpoints of such edge e is already connected with at least another point by a path in G.
Now, we need to prove such a cycle created by adding a new edge is unique. By contradiction, suppose two undirected cycles are created, both of which contain edge e. That means there would be two paths already in existence between two endpoints of edge e. This contradicts condition v. Thus, there is only one undirected cycle created after one new edge is added.

6. (vi → i)
Graph G with n vertices contains no cycles; for any new undirected edge e added, graph T+e has exactly one unoriented cycle.
By contraction, assume that G is not connected; then G has more than one component. Then the addition of a new unoriented edge between two vertices of different components in G would not ascertain the creation one cycle. This contradicts statement vi. Thus, G has to be connected; and since G has no unoriented cycle, G is a tree.

__Reference__: J.L. Gross, J. Yellen, M. Anderson, Graph theory and its applications, CRC press.