**Nagoya University, G30 program** Spring 2020
Mathematics Tutorial IIa ✌ Instructor : Serge Richard

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## Midterm

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**Exercise 1** *Consider the parametric curve in* $\mathbb{R}^3$ *defined by the function* $f : [0, \infty) \to \mathbb{R}^3$ *given by*

$$f(t) = \begin{pmatrix} t \\ t \\ t^{3/2} \end{pmatrix}.$$

*(i) Compute the tangent vector to this curve at any* $t > 0$*, and its norm,*

*(ii) Compute the length of this curve for* $t \in [0, 4]$*.*

**Exercise 2** *Consider the function* $h : \mathbb{R}^3 \to \mathbb{R}$ *given by* $h(x, y, z) = xy^2z^3$*.*

*(i) Compute the partial derivatives of* $h$*, and its gradient,*

*(ii) Compute the directional derivative of* $h$ *in any direction defined by* $V \in \mathbb{R}^3$ *with* $\|V\| = 1$*,*

*(iii) Is* $h$ *differentiable, and why ?*

**Exercise 3** *Consider the function* $g : \mathbb{R}^3 \to \mathbb{R}$ *given by*

$$g(x, y, z) = \mathrm{e}^x - x\cos(y) + \cos(z).$$

*(i) Show that* $\mathbf{0} = (0, 0, 0)$ *is a critical point for* $g$*,*

*(ii) Compute the Hessian matrix for* $g$*,*

*(iii) Study if* $\mathbf{0}$ *is a local maximum, a local minimum, a saddle point, or if we can not conclude by looking at the Hessian matrix only.*

**Exercise 4** *Consider the domain* $\Omega \subset \mathbb{R}_+ \times \mathbb{R}_+$ *defined by the four curves of equation*

$$y = 2x, \quad y = x/2, \quad y = 4/x, \quad y = 1/4x.$$

*Consider also the function* $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}_+$ *given by* $f(x, y) = \big(xy, y/x\big)$*.*

*(i) Represent the domain* $\Omega$*,*

*(ii) Compute the Jacobian matrix of* $f$*, and its determinant,*

*(iii) Represent the image of* $\Omega$ *by* $f$*, namely* $f(\Omega)$*.*

# Midterm

June 2020

## Exercise 1 4 pts

i) $\gamma'(t) = \begin{pmatrix} 1 \\ 1 \\ 3/2 \cdot t^{1/2} \end{pmatrix}$ 1 , $\|\gamma'(t)\| = \left(1 + 1 + \frac{9}{4} t\right)^{1/2}$

$$= \underline{\underline{\left(2 + \frac{9}{4} t\right)^{1/2}}} .$$ 1

ii) $L = \int_0^4 \|\gamma'(t)\| \, dt = \int_0^4 \left(2 + \frac{9}{4} t\right)^{1/2} dt$

$$= \frac{2}{3} \cdot \left(2 + \frac{9}{4} t\right)^{3/2} \frac{4}{9} \Big|_{t=0}^{t=4} = \underline{\underline{\frac{8}{27}\left((11)^{3/2} - (2)^{3/2}\right)}} .$$ 2

## Exercise 2 5 pts

i) $\partial_x h(x,y,z) = y^2 z^3$ , $\partial_y h(x,y,z) = 2xyz^3$
and $\partial_z h(x,y,z) = 3xy^2z^2$.

$$\Rightarrow \nabla h(x,y,z) = \underline{\underline{\begin{pmatrix} y^2 z^3 \\ 2xyz^3 \\ 3xy^2z^2 \end{pmatrix}}} .$$ 2

ii) Since $h$ has continuous partial derivatives, one has $D_v h(x,y,z) = v \cdot \nabla h(x,y,z)$

$$\Rightarrow D_v h(x,y,z) = v_1 y^2 z^3 + 2 v_2 xyz^3 + 3 v_3 xy^2 z^2 .$$

for $v = (v_1, v_2, v_3)$. 2

iii) $h$ is differentiable because its partial derivatives are continuous. 1

Exercise 3 5 pts

i) $\partial_x g(x,y,z) = e^x - \cos(y)$
$\partial_y g(x,y,z) = x \sin(y)$
$\partial_z g(x,y,z) = -\sin(z)$

$$\Rightarrow \nabla g(0,0,0) = \begin{pmatrix} 1-1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$\mathbb{0}$ is a critical point of $g$.

2

ii) $\partial_x \partial_y g(x,y,z) = \sin(y) = \partial_y \partial_x g(x,y,z)$
$\partial_x \partial_z g(x,y,z) = 0 = \partial_z \partial_x g(x,y,z)$
$\partial_y \partial_z g(x,y,z) = 0 = \partial_z \partial_y g(x,y,z)$
$\partial_x \partial_x g(x,y,z) = e^x$
$\partial_y \partial_y g(x,y,z) = x \cos(y)$
$\partial_z \partial_z g(x,y,z) = -\cos(z)$,

$$\Rightarrow Hg(\mathbb{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ with}$$

eigenvalues $(1, 0, -1)$.

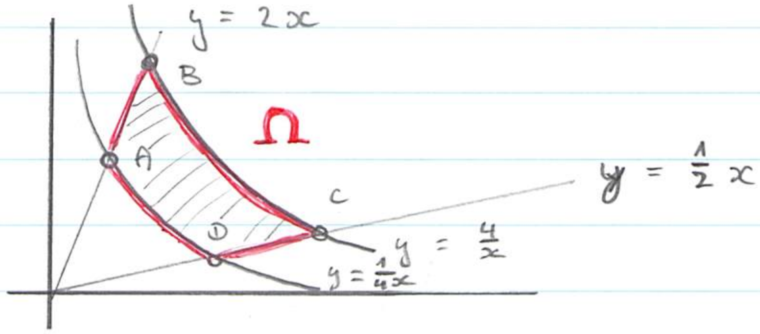
2

iii) Because of the eigenvalue 0, we can not conclude about the nature of this critical point.

1

## Exercise 4 5 pts

i)

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Let us compute A, B, C and D (not strictly required):

A : $\frac{1}{4x} = 2x \iff x^2 = \frac{1}{8}$ , $A = \left(\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

B : $\frac{4}{x} = 2x \iff x^2 = 2$ , $B = (\sqrt{2}, 2\sqrt{2})$

C : $\frac{4}{x} = \frac{1}{2}x \iff x^2 = 8$ , $C = (2\sqrt{2}, \sqrt{2})$

D : $\frac{1}{4x} = \frac{1}{2}x \iff x^2 = \frac{1}{2}$ , $D = \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$.

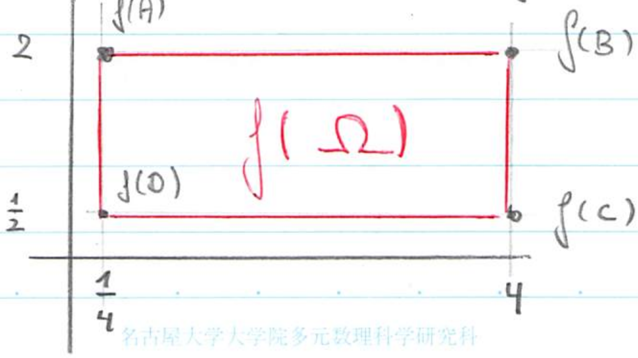
ii) $$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}(x,y) = \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}.$$

$$|J_f(x,y)| = \frac{y}{x} + \frac{y}{x} = \underline{\underline{2\frac{y}{x}}}.$$

2

iii) Line $(x, 2x)$ through $f$ : $f(x,2x) = (2x^2, 2)$
Line $(x, \frac{1}{2}x)$ " " : $f(x, \frac{1}{2}x) = (\frac{1}{2}x^2, \frac{1}{2})$
Curve $(x, \frac{4}{x})$ " " : $f(x, \frac{4}{x}) = (4, \frac{4}{x^2})$
Curve $(x, \frac{1}{4x})$ " " : $f(x, \frac{1}{4x}) = (\frac{1}{4}, \frac{1}{4x^2})$

$f(\Omega)$ :

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