Examples of the arc length parametrization - 2 - Catenary and Cycloid

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May,4, 2020

This is related to

- Chapter 2 in Calculus II
- Chapter 2 in O'Neill

Example 3 (Catenary) Let a and c be positive numbers. Let $f : [-c, c] \to \mathbb{R}^2$ be a parametric curve such that $f(t) = (t, a\cosh(\frac{t}{a}))$ for t in $[-c, c] \subset \mathbb{R}$. f is a function of class C^1 on $(-c, c)$. The corresponding regular curve $f((-c, c))$ is called Catenary (FIG.1). We calculate here the length L_f of the curve in [0,c].

$$L_f := \int_0^c \|f'(t)\|dt = \int_0^c \|(1, \sinh(\frac{t}{a}))\|dt = \int_0^c (1 + (\sinh(\frac{t}{a}))^2)^{1/2}dt$$

(Since $\cosh(\frac{t}{a}) > 0$ on $(0, c)$)

$$= \int_0^c \cosh(\frac{t}{a})dt = a\sinh(\frac{c}{a}).$$

Let $\psi : (0, c) \to \mathbb{R}$ be a function given for any $t \in (0, c)$ by

$$\psi(t) := \int_0^t \|f'(s)\|ds = a\sinh(\frac{t}{a}).$$

ψ is strictly increasing on $[0, c]$ and differentiable (FIG.1), with $\psi'(t) = \|f'(t)\| = \cosh(\frac{t}{a}) > 0$ for any $t \in (0, c)$. ψ has image equal to $[0, L_f]$. It follows that ψ has an inverse $\psi^{-1} : [0, L_f] \to [0, c]$.

In this example,

$$s = \psi(t) = a\sinh(\frac{t}{a}) = a\frac{e^{\frac{t}{a}} - e^{-\frac{t}{a}}}{2},$$
$$\frac{2s}{a} = e^{\frac{t}{a}} - e^{-\frac{t}{a}},$$
$$e^{2\frac{t}{a}} - \frac{2s}{a}e^{\frac{t}{a}} - 1 = 0,$$
$$e^{\frac{t}{a}} = \frac{s}{a} \pm \left(\frac{s^2}{a^2} + 1\right)^{1/2} = \frac{s \pm \sqrt{s^2 + a^2}}{a}.$$

Since $e^{\frac{t}{a}} > 0$, the right hand side must be $\frac{s + \sqrt{s^2 + a^2}}{a}$.
Hence $\frac{t}{a} = \ln\left(\frac{s + \sqrt{s^2 + a^2}}{a}\right)$. That is, $t = a\ln\left(\frac{s + \sqrt{s^2 + a^2}}{a}\right)$.
Thus, if we set

$$\varphi : [0, L_f] \to [0, c], \ \ \varphi(s) := \psi^{-1}(s) = a\ln\left(\frac{s + \sqrt{s^2 + a^2}}{a}\right),$$

then φ is a diffeomorphism of class C^1 on $(0, L_f)$, and the composed map $f \circ \varphi : [0, L_f] \to \mathbb{R}^2$ is

$$(f \circ \varphi)(s) = f(\varphi(s)) = \left(a\ln(\frac{s + \sqrt{s^2 + a^2}}{a}), \ \ a\cosh(\ln(\frac{s + \sqrt{s^2 + a^2}}{a}))\right).$$

This satisfies

$$(f \circ \varphi)'(s) = f'(\varphi(s))\varphi'(s) = f'(\varphi(s))\frac{1}{\|f'(\psi^{-1}(s))\|},$$

with norm

$$\|(f \circ \varphi)'(s)\| = \left\|f'(\varphi(s))\frac{1}{\|f'(\varphi(s))\|}\right\| = \frac{1}{\|f'(\varphi(s))\|}\|f'(\varphi(s))\| = 1.$$

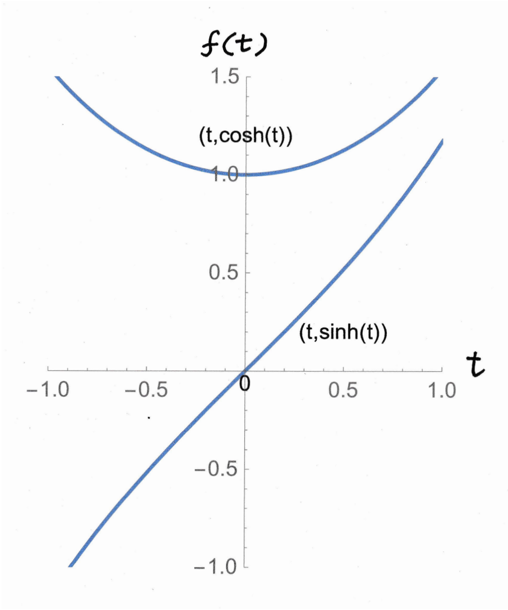

FIG. 1 Graphs of cosh(t) (Catenary) and sinh(t) for $a = 1$.

$\square$

Example 4 (Cycloid) Let r be a positive number. Let $f : [0, 2\pi] \to \mathbb{R}^2$ be a parametric curve such that $f(t) = (-r\sin(t) + rt, -r\cos(t) + r)$ for t in $[0, 2\pi]$. f is a function of class C^1 on $(0, 2\pi)$. The corresponding regular curve $f((0, 2\pi))$ is called Cycloid (FIG.2).

$$f'(t) = (-r\cos(t) + r, r\sin(t)),$$

$$\|f'(t)\| = \sqrt{(-r\cos(t) + r)^2 + (r\sin(t))^2} = \sqrt{2r^2 - 2r^2\cos(t)} = \sqrt{2}r\sqrt{1 - \cos(t)}$$

$$= \sqrt{2}r\sqrt{1 - (2\cos^2(\frac{t}{2}) - 1)} = \sqrt{2}r\sqrt{2 - 2\cos^2(\frac{t}{2})} = 2r\sqrt{1 - \cos^2(\frac{t}{2})}$$

(Since $\sin(\frac{t}{2}) > 0$ on $(0, 2\pi)$)

$$= 2r\sin(\frac{t}{2}).$$

The length L_f of the curve is

$$L_f := \int_0^{2\pi} \|f'(t)\|dt = \int_0^{2\pi} 2r\sin(\frac{t}{2})dt = (-4r)\cos(\frac{t}{2})\Big|_{t=0}^{t=2\pi} = (-4r)(-1 - 1) = 8r.$$

Let $\psi : (0, 2\pi) \to \mathbb{R}$ be a function given for any $t \in (0, 2\pi)$ by

$$\psi(t) := \int_0^t \|f'(s)\|ds = (-4r)\cos(\frac{s}{2})\Big|_{s=0}^{t} = (-4r)\cos(\frac{t}{2}) + 4r.$$

ψ is strictly increasing on $[0, 2\pi]$ and differentiable, with

$$\psi'(t) = \|f'(t)\| = 2r\sin(\frac{t}{2}) > 0 \ \ \forall t \in (0, 2\pi).$$

ψ has image equal to $[0, L_f]$. It follows that ψ has an inverse $\psi^{-1} : [0, L_f] \to [0, 2\pi]$.

In this example,

$$\begin{aligned} s = \psi(t) &= (-4r)\cos(\frac{t}{2}) + 4r, \\ \cos(\frac{t}{2}) &= \frac{s-4r}{-4r} = 1 - \frac{s}{4r}, \\ \frac{t}{2} &= \arccos(1 - \frac{s}{4r}), \\ t = \psi^{-1}(s) &= 2\arccos(1 - \frac{s}{4r}) \end{aligned}$$

Thus, if we set

$$\varphi : [0, L_f] \to [0, 2\pi], \ \ \varphi(s) := \psi^{-1}(s) = 2\arccos(1 - \frac{s}{4r}),$$

then φ is a diffeomorphism of class C^1 on $(0, L_f)$ (FIG.3). The composed map $f \circ \varphi : [0, L_f] \to \mathbb{R}^2$ is

$$(f \circ \varphi)(s) = f(\varphi(s)) = (-r\sin(\varphi(s)) + r\varphi(s), -r\cos(\varphi(s)) + r)$$
$$= \left(-r\sin(2\arccos(1 - \frac{s}{4r})) + r(2\arccos(1 - \frac{s}{4r})), -r\cos((2\arccos(1 - \frac{s}{4r})) + r\right).$$

This satisfies

$$(f \circ \varphi)'(s) = f'(\varphi(s))\varphi'(s) = f'(\varphi(s))\frac{1}{\|f'(\psi^{-1}(s))\|},$$

with norm

$$\|(f \circ \varphi)'(s)\| = \left\| f'(\varphi(s))\frac{1}{\|f'(\varphi(s))\|} \right\| = \frac{1}{\|f'(\varphi(s))\|}\|f'(\varphi(s))\| = 1.$$

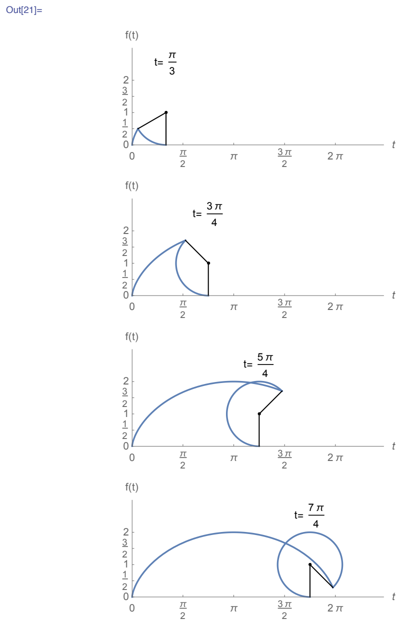

FIG. 2 drawing a Cycloid for r=1.

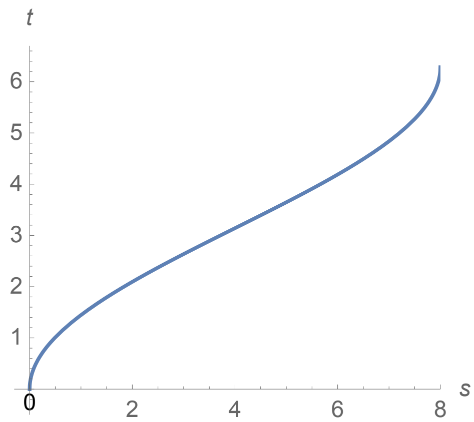

FIG. 3 t(s)=2 arccos(1-s/4), when r=1.

□

References

[1] Wikipedia, Arc length.

[2] Mathematics Stack Exchange, "Need help to parametrize the catenary by arc length".

[3] wwwf.imperial.ac.uk/metric/ ..., "Hyperbolic Functions: Inverses".

[4] Barrett O'Neill, Elementary Differential Geometry, Elsevier 2006.