Summary : Chap I – IV

C*-algebras :

- C* - property : $\|a^*a\| = \|a\|^2 \quad \forall a$.
- GNS construction : any C* -algebra can be represented faithfully in $\mathcal{B}(\mathcal{H})$.
- Quotient : If $\mathcal{J} \subset \mathcal{C}$ is an ideal in a C*-alg., then $\mathcal{C}/\mathcal{J}$ is a C*-alg.
- Smallest unitization $\tilde{\mathcal{C}} = \mathcal{C} + \mathbb{C}$.
- Elements of spectral theory : invertibility, spectrum, normal op., self-adjoint, unitary, projection, isometries, partial isometries, positivity, bounded functional calculus for normal elements : $f(a)$, spectral mapping thm $\sigma(f(a)) = f(\sigma(a))$.
- Lifting : If $\mathcal{C} \to \underset{\substack{\cup\\ b}}{Q}$ surjective, what about $\varphi^{-1}(b)$?

About projections and K_0 :

Equivalence relations : $p \sim_h q$, $p \sim q$,

(homotopy) (Murray-von Neumann)

$p \sim_u q$: all equivalent in $M_n(\mathcal{A})$: $\sim_0$

(unitary)

$K_0(\mathcal{A})$ "$\approx$" $\{$ equivalence class of projections $\}$

want to have more structures

1° $p \oplus q$ in $\mathcal{P}_\infty(\mathcal{A}) = \bigcup_n \mathcal{P}_n(\mathcal{A})$

$\rightsquigarrow$ Abelian semi-group

$\to$ Grothendieck construction : Abelian group

$K_0(\mathcal{A})$: $p \in \mathcal{P}_\infty(\mathcal{A}) \to [p]_0 = \gamma([p]_{\sim_0})$

Grothendieck map

2° If $\varphi : \mathcal{A} \to Q$ $*$-homomorphism , then

group morphism

$K_0(\varphi) : K_0(\mathcal{A}) \to K_0(Q)$, with

$K_0(\varphi)([p]_0) = [\varphi(p)]_0$

⚠ If $\mathcal{A}$ not unital , this funtor is not even half-exact $\Rightarrow$ we keep it for unital algebras only.

If $\mathcal{C}$ is not unital, we use

$$0 \to \mathcal{C} \to \tilde{\mathcal{C}} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0$$

and set $K_0(\mathcal{C}) := \ker(K_0(\pi))$ ...

We restore half-exactness, get split-exactness.

<u>About unitary elements</u>:

<u>Equivalence relation</u> : $u \sim_h v$

normal subgroup of $\mathcal{U}(\mathcal{C})$

<u>In a unital C^*-algebra</u> : $\mathcal{U}_0(\mathcal{C}) \ni \{e^{ia} \mid a = a^*\}$
$\cup \{u \mid \sigma(u) \neq \mathbb{T}\}$

In $M_2(\mathcal{C})$:

$$\begin{pmatrix} u & \\ & v \end{pmatrix} \sim_h \begin{pmatrix} v & \\ & u \end{pmatrix} \sim_h \begin{pmatrix} uv & \\ & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & \\ & 1 \end{pmatrix}$$

For invertible elements : $a = \underset{\uparrow \text{unitary}}{u}|a| = u(a^*a)^{1/2}$ ← polar decomposition.

and $GL(\mathcal{C}) \ni a \longmapsto u = u(a) \in \mathcal{U}(\mathcal{C})$

continuous in $GL(\mathcal{C})$

- If $a \sim_h b$ in $GL(\mathcal{C})$, then

$u(a) \sim_h u(b)$ in $\mathcal{U}(\mathcal{C})$.