K-theory 3

Kato Shinya

Proof of Proposition 3.2.5 refered to [RLL, Proposition 3.1.8]

Proposition 3.2.5 (Universal property of K_0).

Let C be a unital C^*-algebra, and let H be an Abelian group.
Suppose that there exists $\nu : \mathcal{P}_\infty(C) \to H$ satisfying the three conditions:

(i) $\nu(p \oplus q) = \nu(p) + \nu(q)$ for any $p, q \in \mathcal{P}_\infty(C)$,

(ii) $\nu(0_C) = 0$,

(iii) If $p, q \in \mathcal{P}_n(C)$ for some $n \in \mathbb{N}$ and if $p \sim_h q \in \mathcal{P}_n(C)$, then $\nu(p) = \nu(q)$.

Then there exists a unique group homomorphism $\alpha : K_0(C) \to H$ such that the diagram

$$\begin{tikzcd} \mathcal{P}_\infty(C) \arrow[d, "{[\cdot]_0}"'] \arrow[dr, "\nu"] & \\ K_0(C) \arrow[r, "\alpha"'] & H \end{tikzcd}$$

is commutative.

proof) First, we show that if $p \sim_0 q$, then $\nu(p) = \nu(q)$ for $p, q \in \mathcal{P}_\infty(C)$.
Find $l, m \in \mathbb{N}$ such that $p \in \mathcal{P}_l(C)$ and $q \in \mathcal{P}_m(C)$.
Let n be an integer greater than both l and m and define
$p' = p \oplus 0_{n-l}$ and $q' = q \oplus 0_{n-m}$.
$p', q' \in \mathcal{P}_n(C)$ and $p' \sim_0 p \sim_0 q \sim_0 q'$.
$\because$ The first and third equivalences follow from Proposition 3.1.1 (i), the second follows from assumption.
This implies $p' \sim_0 q'$ that is $p' \sim q'$ ($p, q \in \mathcal{P}_n(C)$).
By Proposition 2.2.10 (i), (ii),

$$\begin{pmatrix} p' & 0 & & \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} \sim_h \begin{pmatrix} q' & 0 & & \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} \text{ in } \mathcal{P}(M_4(M_n(C))) = \mathcal{P}_{4n}(C).$$

By assumption,

$$\nu(p) = \nu(p) + \overbrace{\nu(0) + \cdots + \nu(0)}^{4n-l} = \nu(p' \oplus 0_{3n}) = \nu(q' \oplus 0_{3n}) = \nu(q).$$

Kato Shinya

It follows that the map $\beta : \mathcal{D}(\mathcal{C}) \to H$ defined by $\beta([p]_{\mathcal{D}}) = \nu(p)$ is well-defined.

Additivity of β is

$$\beta([p]_{\mathcal{D}} + [q]_{\mathcal{D}}) = \beta([p \oplus q]_{\mathcal{D}}) = \nu(p \oplus q) = \nu(p) + \nu(q) = \beta([p]_{\mathcal{D}}) + \beta([q]_{\mathcal{D}}).$$

By Proposition 3.2.1, there is a unique group homomorphism α such that

$$\begin{tikzcd} \mathcal{D}(\mathcal{C}) \arrow[r, "\beta"] \arrow[d, "\gamma_{\mathcal{D}}"'] & H \\ K_0(\mathcal{C}) = \mathcal{G}(\mathcal{D}(\mathcal{C})) \arrow[ur, "\alpha"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] & \end{tikzcd}$$

is commutative.

The uniqueness of α follows from Proposition 3.2.4., (3.2). $\square$

The universal property of Grothendieck group. (Proposition 3.2.1 (i))

Let $\mathcal{D}$ be an Abelian semigroup and H be an Abelian group and $\varphi : \mathcal{D} \to H$ be an additive map.

Then there exists a unique group homomorphism $\psi : \mathcal{G}(\mathcal{D}) \to H$ such that

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\varphi"] \arrow[d, "\gamma_{\mathcal{D}}"'] & H \\ \mathcal{G}(\mathcal{D}) \arrow[ur, "\psi"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] & \end{tikzcd}$$

is commutative.

(:) We construct ψ by $\psi(\langle x, y \rangle) = \varphi(x) - \varphi(y)$.

If $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$, then $\exists z \in \mathcal{D}$, $x_1 + y_2 + z = y_1 + x_2 + z$ and

$\varphi(x_1) + \varphi(y_2) + \varphi(z) = \varphi(y_1) + \varphi(x_2) + \varphi(z)$.

Since H is an Abelian group, we obtain $\varphi(x_1) - \varphi(y_1) = \varphi(x_2) - \varphi(y_2)$.

Hence ψ is well-defined.

$\psi \circ \gamma_{\mathcal{D}}(x) = \psi(\langle x+y, y \rangle) = \varphi(x+y) - \varphi(y) = \varphi(x) + \varphi(y) - \varphi(y) = \varphi(x)$

Therefore the diagram is commutative.

The uniqueness of ψ follows from that $\mathcal{G}(\mathcal{D})$ is determined by $\mathcal{D}$. $\square$