

Exercise 3.4.4

Provide the proofs for the statements of Example 3.4.1, 3.4.2 and 3.4.3.

Example 3.4.1

For any $n \in \mathbb{N}$, one has $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$.

In fact, if tr denotes the usual trace already introduced in Exercise 3.1.4, then $K_0(\text{tr}) : K_0(M_n(\mathbb{C})) \rightarrow \mathbb{Z}$ is an isomorphism.

First, we introduce a group homomorphism $K_0(\text{tr}) : K_0(M_n(\mathbb{C})) \rightarrow \mathbb{Z}$ such that the diagram

$$\begin{array}{ccc} P_{\infty}(M_n(\mathbb{C})) & \xrightarrow{\text{tr}} & \mathbb{Z} \\ \downarrow [\cdot]_0 & \circlearrowleft & \uparrow K_0(\text{tr}) \\ K_0(M_n(\mathbb{C})) & & \end{array}$$

commutative. (see my other report)

By Proposition 3.2.5, we should check that $\text{tr} : P_{\infty}(M_n(\mathbb{C})) \rightarrow \mathbb{Z}$

satisfies the three conditions: (we identify $M_m(M_n(\mathbb{C})) \cong M_{mn}(\mathbb{C})$ canonically.)

(i) $\text{tr}(p \oplus q) = \text{tr}(p) + \text{tr}(q)$ for any $p, q \in P_{\infty}(M_n(\mathbb{C}))$;

(ii) $\text{tr}(0) = 0$;

(iii) If $p, q \in P_m(M_n(\mathbb{C}))$ for some $m \in \mathbb{N}$, and if $p \sim_n q \in P_m(M_n(\mathbb{C}))$, then $\text{tr}(p) = \text{tr}(q)$.

☺ (i), (ii) are clear by definition of tr .

(iii) Suppose that $p, q \in P_m(M_n(\mathbb{C}))$ for some $m \in \mathbb{N}$.

Then we can consider $p, q \in P(M_{mn}(\mathbb{C}))$.

By Lemma 2.2.9, if $p \sim_n q \in P(M_{mn}(\mathbb{C}))$, then

$p \sim_u q$, i.e., $\exists u \in U(M_{mn}(\mathbb{C}))$ s.t. $q = upu^*$.

Hence $\text{tr}(q) = \text{tr}(upu^*) = \text{tr}(u^*up) = \text{tr}(p)$.

☺ tr has the trace property $\text{tr}(ab) = \text{tr}(ba)$, $a, b \in M_n(\mathbb{C})$

Therefore $K_0(\text{tr}) : K_0(M_n(\mathbb{C})) \rightarrow \mathbb{Z}$ is the group homomorphism which satisfies $K_0(\text{tr})([p]_0) = \text{tr}(p)$ ($p \in \mathcal{P}_0(M_n(\mathbb{C}))$) by Proposition 3.2.5.

Next, we show that $K_0(\text{tr})$ is an isomorphism.

Let x be in $K_0(M_n(\mathbb{C}))$. By Proposition 3.2.4, there exists $p, q \in \mathcal{P}_m(M_n(\mathbb{C}))$ such that $x = [p]_0 - [q]_0$.

$K_0(\text{tr})(x) = 0$ implies that $\text{tr}(p) = \text{tr}(q)$.

By Exercise 3.1.4. (see Mr. Narita's report),

$\text{tr}(p) = \text{tr}(q)$ is equivalent to $p \sim q$.

Hence $x = [p]_0 - [q]_0 = 0$ and $K_0(\text{tr})$ is injective.

We already show that $K_0(\text{tr})$ is a group homomorphism.

So we only have to show that 1 is in the image of $K_0(\text{tr})$ due to show that $K_0(\text{tr})$ is surjective.

Take $p' = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \in M_n(\mathbb{C})$, the matrix elements are $p'_{11} = 1$ and $p'_{ij} = 0$ (otherwise).

Clearly $p' \in \mathcal{P}(M_n(\mathbb{C}))$ and $\text{tr}(p') = 1$.

Hence $K_0(\text{tr})([p']_0) = \text{tr}(p') = 1$ and $K_0(\text{tr})$ is surjective.

Therefore $K_0(\text{tr})$ is an isomorphism from $K_0(M_n(\mathbb{C}))$ to \mathbb{Z} . \square

Example 3.4.2

If \mathcal{H} is an infinite dimensional separable Hilbert space,
then we have $K_0(\mathcal{B}(\mathcal{H})) = \{0\}$.

By Exercise 3.1.5 (see my other report), $D(\mathcal{B}(\mathcal{H})) \cong \mathbb{Z} + \mathbb{U}\{\infty\}$.

So we construct the Grothendieck group of $\mathbb{Z} + \mathbb{U}\{\infty\}$.

We define equivalence relation by

$(x_1, y_1) \sim (x_2, y_2)$ if there exists $z \in \mathbb{Z} + \mathbb{U}\{\infty\}$ such that
 $x_1 + y_2 + z = x_2 + y_1 + z$.

Now, if we take $\infty \in \mathbb{Z} + \mathbb{U}\{\infty\}$, then $x_1 + y_2 + \infty = x_2 + y_1 + \infty = \infty$

for any $x_1, y_1, x_2, y_2 \in \mathbb{Z} + \mathbb{U}\{\infty\}$. (addition rule is $n + \infty = \infty, \forall n \in \mathbb{Z} + \mathbb{U}\{\infty\}$.)

So any pair $(x, y) \in (\mathbb{Z} + \mathbb{U}\{\infty\}) \times (\mathbb{Z} + \mathbb{U}\{\infty\})$ is equivalent to $(0, 0)$

and $(\mathbb{Z} + \mathbb{U}\{\infty\}) \times (\mathbb{Z} + \mathbb{U}\{\infty\}) / \sim \cong \{0\}$ as the additive groups.

Hence $K_0(\mathcal{B}(\mathcal{H})) = \{0\}$. \square

the Hilbert space

According to RLL, Example 3.3.3, \mathcal{H} no need to be separable.

If \mathcal{H} is not separable, then $K_0(\mathcal{B}(\mathcal{H}))$ is also equal to $\{0\}$.

Example 3.4.3

If Ω is a compact, connected and Hausdorff space,
then there exists a surjective group morphism

$$\dim : K_0(C(\Omega)) \rightarrow \mathbb{Z}$$

which satisfies for $p \in \mathcal{P}_0(C(\Omega))$ and $x \in \Omega$, $\dim([p]_0) = \text{tr}(p(x))$.

We show that $\text{tr}(p(x))$ is independent to x firstly. For $p \in \mathcal{P}_0(C(\Omega))$,
The map $\Omega \ni x \mapsto \text{tr}(p(x)) \in \mathbb{Z}$ is constant since Ω is connected.
(If Ω is connected, every function in $C(\Omega, \mathbb{Z})$ is constant.)

Therefore $\text{tr}(p(x))$ is independent to x .

Next, we define the map $\tau_x : C(\Omega) \rightarrow \mathbb{C}$ by $\tau_x(f) = f(x)$.

It is a trace on $C(\Omega)$ since $\tau_x(fg) = f(x)g(x) = g(x)f(x) = \tau_x(gf)$, $\forall f, g \in C(\Omega)$.

We can extend τ_x to $\tau_x : M_n(C(\Omega)) \rightarrow M_n(\mathbb{C})$ for any $n \in \mathbb{N}$.

If $p \in \mathcal{P}(M_n(C(\Omega)))$, then $\tau_x(p) \in \mathcal{P}(M_n(\mathbb{C}))$ and

$[\tau_x(p)]_0$ is determined by $\text{tr}(\tau_x(p)) = \text{tr}(p(x))$.

Hence the induced map $K_0(\tau_x) : K_0(C(\Omega)) \rightarrow \mathbb{Z}$ satisfies

$K_0(\tau_x)([p]_0) = [\tau_x(p)]_0$ and we define $\dim = K_0(\tau_x)$.

The unit $1 \in C(X)$ is a projection and

$$\text{tr} \begin{pmatrix} \tau_x & 0 \\ 0 & 0 \end{pmatrix} = 1.$$

Hence $\dim([1]_0) = 1$ and \mathbb{Z} has a unit.

Therefore \dim is surjective. \square

I referred [RLL, Example 3.3.5].