K-theory

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Exercise 3.1.5

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and let p, q be projections in $B(\mathcal{H})$.

(i) Show that $p \sim q$ if and only if $\dim(p\mathcal{H}) = \dim(q\mathcal{H})$,

(ii) Show that $p \sim_u q$ if and only if $\dim(p\mathcal{H}) = \dim(q\mathcal{H})$ and $\dim(p\mathcal{H}^\perp) = \dim(q\mathcal{H}^\perp)$,

(iii) Infer that $D(B(\mathcal{H})) \simeq \mathbb{Z}_+ \cup \{\infty\} \equiv$ where $\{0, 1, 2, \cdots, \infty\}$, where the usual addition on $\mathbb{Z}_+$ is considered together with the addition $n + \infty = \infty + n = \infty$ for all $n \in \mathbb{Z}_+ \cup \{\infty\}$.

(i) ($\Rightarrow$) Suppose that $p \sim q$, i.e., there exists $v \in B(\mathcal{H})$ such that $p = v^*v$ and $q = vv^*$.

- Assume that $\dim(p\mathcal{H}) = n$.
  By $q = q^2 = vv^*vv^* = vpv^*$, $q\mathcal{H} = vpv^*\mathcal{H}$.
  By assumption, $\dim(vpv^*\mathcal{H}) \leq n$. Hence $\dim(q\mathcal{H}) \leq n$.
  If $\dim(q\mathcal{H}) < n$, then we obtain $\dim(p\mathcal{H}) < n$ similarly. ($p = v^*qv$)
  It is contradiction and $\dim(q\mathcal{H}) = n = \dim(p\mathcal{H})$.
- Assume that $\dim(p\mathcal{H}) = \infty$.
  If $\dim(q\mathcal{H}) < \infty$, then $\dim(p\mathcal{H}) < \infty$ since above argument.
  Hence $\dim(q\mathcal{H}) = \infty$.

($\Leftarrow$) Suppose that $\dim(p\mathcal{H}) = \dim(q\mathcal{H})$.
Take the basis of $p\mathcal{H}$, write $\{e_i\}_{i \in I}$ ($I = \mathbb{N}$ or $\{1, \cdots, n\}$), and the basis of $q\mathcal{H}$, $\{f_i\}_{i \in I}$.

We define $v \in B(\mathcal{H})$ by
$$v(e_i) = f_i \text{ and } v(x) = 0 \quad (x \in p\mathcal{H}^\perp).$$
Clearly $v^*(f_i) = e_i$ and $v^*(y) = 0$ $(y \in q\mathcal{H}^\perp)$.

$$\therefore v^*v(x) = \begin{cases} x & x \in p\mathcal{H} \\ 0 & x \in p\mathcal{H}^\perp \end{cases}, \quad vv^*(x) = \begin{cases} x & x \in q\mathcal{H} \\ 0 & x \in q\mathcal{H}^\perp \end{cases}.$$

$\therefore p = v^*v$, $q = vv^*$. $\square$

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(ii) By Lemma 2.2.4, $p \sim_u q$ if and only if $p \sim q$ and $1-p \sim 1-q$.
Furthermore, $(1-p)\mathcal{H} = p\mathcal{H}^{\perp}$ and $(1-q)\mathcal{H} = q\mathcal{H}^{\perp}$.
So from (i),

$$p \sim_u q \iff p \sim q \text{ and } 1-p \sim 1-q$$
$$\iff \dim(p\mathcal{H}) = \dim(q\mathcal{H}) \text{ and } \dim((1-p)\mathcal{H}) = \dim((1-q)\mathcal{H})$$

□.

(iii) If $\mathcal{H}$ is separable infinite dim, then $\mathcal{H} \cong \mathcal{H}^n$ and $B(\mathcal{H}^n) \cong M_n(B(\mathcal{H}))$.
For $p \in P_n(B(\mathcal{H}))$, $[p]_D$ is determined by the number of $\dim(p\mathcal{H}^n)$.
Moreover $p' = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \end{pmatrix} \in P_\infty(B(\mathcal{H}))$ is equivalent to p. (The size of $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ is $\dim(p\mathcal{H}^n)$.)
By the well-definedness of $[\cdot]_D$,
for $p, q \in P_\infty(B(\mathcal{H}))$, we can take $p' = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \sim_0 p$ and $q' = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \sim_0 q$,
and

$$[p]_D + [q]_D = [p']_D + [q']_D = \left[\begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 0 \end{pmatrix} \right]_D .$$

Therefore the map $F : D(B(\mathcal{H})) \to \mathbb{Z}_+ \cup \{\infty\}$ defined by $F([p]_D) = \dim(p\mathcal{H}^n)$ is isomorphism of semigroups. □