Hall's marriage theorem

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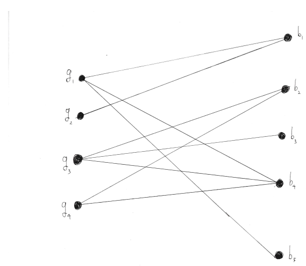
The marriage theorem, proved in 1935 by Philip Hall, answers the following question, known as the marriage problem: if there is a finite set of girls, each of whom knows several boys. Under what conditions can all the girls marry the boys in such a way that each girl marries a boy she knows? For example, if there are four girls $\{g_1, g_2, g_3, g_4\}$ and five boys $\{b_1, b_2, b_3, b_4, b_5\}$, and the friendship are shown below, then a possible solution is for g_1 to marry b_4, g_2 marry b_1, g_3 to marry b_3, and g_4 to marry b_2.



Theorem: A necessary and sufficient condition for a solution of the marrige problem is that each set of k girls collectively knows at least k boys, for
$1 \le k \le m$.

Proof. Let the girls be $g_1, g_2, g_3, \ldots, g_n$, and the boys be $b_1, b_2, b_3, \ldots, b_m$, with $m \ge n$.

The relation between a girl and a boy and whether they know each other constructs a bipartite graph $V = V_1 \cup V_2$ where V_1 is the set of vertices $g_1, g_2, \ldots, g_n$, V_2 is the set of vertices $b_1, b_2, \ldots, b_m$, and there exists an edge of G between g_i and b_j if the girl g_i knows the boy b_j. In this way, we construct a simple bipartite graph.

Let A is a subset of V_1. Denote $P(A)$ for the subset of V_2 that all the edges from A to V_2 have an endpoint in $P(A)$, and each vertex in $P(A)$ has an edge connects to a vertex in A .

Let $|A|$ represents the number of elements contained subset A.Then, proving the theorem is equivalent to proving that if any A subset of V_1 satisfies $|A| \leq |P(A)|$, then there is a complete matching from V_1 to V_2, namely any g_i is connected with a different b_j.

Thus, assume that

$$|A| \leq |P(A)|$$

for any subset A of V_1.

Adjoining to G a vertex of v adjacent to (and only to) every vertex in V_1 and a vertex w adjacent to (and only to) every vertex in V_2.

Menger's theorem says that if S is a vw-separating set, then $|S| \geq \#$ internaly disjoint path form v to w. (here, # means "the number of")

$|V_1| \geq$ #internally disjoint path from v to w because V_1 is a vw-separating set.

Let $S = A \cup B$ be a v-w separator in which A is subset of V_1 and B is subset of V_2. Then $|S| \geq$ #internally disjoint path from v to w by menger's theorem.

$(V_1 - A)$ and $(V_2 - B)$ are not connected with each other by any edge since if they are connected by some edges to each other, then $A \cup B = S$ is not a separator of v-w. Thus, $P(V_1 - A)$ is a subset of B since $V_1 - A$ is connected with vertices in V_2 but not $V_2 - B$ as we argued above.

With our assumption, we obtain that

$$|V_1 - A| \leq |P(V_1 - A)| \leq |B|$$

As $|S| = |A| + |B|$ (A subset of V_1 and B subset of V_2 are disjoint),

$$|S| \geq |A| + |V_1 - A| = |V_1|$$

From above relations, we have

$$\begin{aligned} |V_1| &\geq \text{\#internally disjoint path from v to w} \\ |S| &\geq \text{\#internaly sidjoint path from v to w by menger's theorem.} \\ |S| &\geq |V_1| \end{aligned}$$

Thus, we get the total relation:

$$|S| \geq |V_1| \geq \text{ \#internally disjoint path from } v \text{ to } w$$

But the menger's theorem says that $|S|_{min} =$ max(#internally disjoint path from v to w)

Thus, $|S|_{min} = |V_1| = \max($ # internally disjoint path from v to w).

Therefore, $|V_1| = max(\#$internally disjoint path from v to w). This means there exists a set of $|V_1|$ internally disjoint paths from v to w in which each path includes a different vertex in V_1 and a different vertex in V_2 from any other path in order to satisfy the internally disjoint condition. Thus, we have the perfect matching from this set of internally disjoint paths. $\square$